

UNIFORM DESCRIPTION OF THE RIGGED CONFIGURATION BIJECTION

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ABSTRACT. We give a uniform description of the bijection Φ from rigged configurations to tensor products of Kirillov–Reshetikhin crystals of the form $\bigotimes_{i=1}^N B^{r_i,1}$ in dual untwisted types: simply-laced types and types $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $E_6^{(2)}$, and $D_4^{(3)}$. We give a uniform proof that Φ is a bijection and preserves statistics. We describe Φ uniformly using virtual crystals for all remaining types, but our proofs are type-specific. We also give a uniform proof that Φ is a bijection for $\bigotimes_{i=1}^N B^{r_i,s_i}$ when r_i , for all i , map to 0 under an automorphism of the Dynkin diagram. Furthermore, we give a description of the Kirillov–Reshetikhin crystals $B^{r,1}$ using tableaux of a fixed height k_r depending on r in all affine types. Additionally, we are able to describe crystals $B^{r,s}$ using $k_r \times s$ shaped tableaux that are conjecturally the crystal basis for Kirillov–Reshetikhin modules for various nodes r .

1. INTRODUCTION

Kashiwara began the study of crystals in the early 1990’s as a method to explore the representation theory of quantum groups [Kas90, Kas91]. One particular application is the highest weight elements of a tensor product of Kirillov–Reshetikhin (KR) crystals naturally index solutions on two-dimensional solvable lattice models from using Baxter’s corner transfer matrix [Bax89]. Kerov, Kirillov, and Reshetikhin introduced combinatorial objects called rigged configurations that naturally index solutions to the Bethe Ansatz for the isotropic Heisenberg spin model [KKR86, KR86]. The row-to-row transfer matrix and the Hamiltonian of the spin chain commute and can be simultaneously diagonalized, which suggests a link between rigged configurations and highest weight elements of a tensor product of KR crystals. This was formalized by Kerov, Kirillov, and Reshetikhin by giving a bijection Φ .

More specifically, Kerov, Kirillov, and Reshetikhin gave a bijection Φ for the tensor product $(B^{1,1})^{\otimes N}$ in type $A_n^{(1)}$ and the corresponding rigged configurations. This was extended to the general case $\bigotimes_{i=1}^N B^{r_i,s_i}$ in type $A_n^{(1)}$ in [KSS02], and it was conjectured that there exists an analogous bijection in other types. For the remaining non-exceptional types, such a bijection for $(B^{1,1})^{\otimes N}$ was given in [OSS03a] and type $E_6^{(1)}$ in [OS12]. Many other special cases are also known: the $\bigotimes_{i=1}^N B^{1,s_i}$ case for non-exceptional types [OSS03b, SS06a]; the $\bigotimes_{i=1}^N B^{r_i,1}$ case for types $D_{n+1}^{(2)}$, $A_{2n}^{(2)}$, and $C_n^{(1)}$ [OSS03c], and type $D_n^{(1)}$ [Sch05]; the case $B^{r,s}$ for type $D_n^{(1)}$ [OSS13] and other non-exceptional types [SS15b]; and both type of tensor products are known for type $D_4^{(3)}$ [Scr16]. Recently, the general case for type $D_n^{(1)}$ was proven [OSSS16]. Additionally, the bijection Φ was extended to a crystal isomorphism for the full crystal in type $A_n^{(1)}$ in [DS06, SW10] and a classical crystal isomorphism for type $D_n^{(1)}$ in [Sak14] and $A_{2n-1}^{(2)}$ in [SS15b].

Despite being defined recursively, obfuscating many of its properties, the bijection Φ has many remarkable (conjectured) attributes. There is a natural statistic defined on tensor products of KR crystals called energy that arose from the related statistical mechanics, but energy is an algebraic statistic whose computation requires using the very intricate combinatorial R -matrix. On the rigged configuration side, there is a combinatorial statistic called cocharge, which also comes from the related physics, and Φ sends energy to cocharge (with interchanging riggings and corigings). This gives a combinatorial proof the $X = M$ conjecture of [HKO+99, HKO+02b]. We recall that the X side comes from the sum over the classically highest weight elements of tensor products of KR crystals and is related to the one-point function of 2D lattice models, and the M side is summed over highest weight rigged configurations and is related to solutions to

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the Bethe equation of the Heisenberg spin chain. Moreover, the combinatorial R -matrix gets sent to the identity map under Φ .

Because of these properties, rigged configurations in type $A_n^{(1)}$ describe the action-angle variables of box-ball systems [KOS⁺06], an ultradiscrete version of the Korteweg-de Vries (KdV) equation. More specifically, the partition $\nu^{(1)}$ describes the sizes of the solitons when there is no interaction [KOS⁺06, Tak05]. A tropicalization of a ratio of (cylindric) loop Schur functions is conjectured to describe Φ for box-ball systems [LPS16, Scr17], and Φ^{-1} can be described using the τ function from the Kadomtsev–Petviashvili (KP) hierarchy [KSY07]. It is also conjectured that rigged configurations are connected to the generalizations of box-ball systems, soliton cellular automata [HKO⁺02a, bM12a, MOW12, Yam04, Yam07].

There are many properties of rigged configurations that are known to be uniform. A crystal structure on rigged configurations was first given for simply-laced types in [Sch06], which was then extended to a classical crystal structure for $U'_q(\mathfrak{g})$ -crystals for affine types [SS15b] and highest weight crystals and $B(\infty)$ for general Kac–Moody algebras in [SS15a, SS17]. Furthermore, the $*$ -involution on $B(\infty)$ [Kas93, Lus90] is the map that replaces all riggings with their respective corrigings [SS16b]. In [SS16a], the bijection Φ was also extended (uniformly) to describe a bijection between the rigged configurations and marginally large tableaux [Cli98, HL08, HL12] for $B(\infty)$.

Similarly, there are also uniform descriptions of KR crystals of the form $B = \bigotimes_{i=1}^N B^{r_i,1}$ using the alcove path model (up to non-dual Demazure arrows) [LL15a] and quantum and projected level-zero LS paths [LNS⁺15, LNS⁺16a, LNS⁺16b, NS06, NS08a, NS08b]. This is based upon the work of Kashiwara [Kas02], where B is a crystal basis of the tensor product of the corresponding KR modules and is constructed from projecting the crystal basis of a level-zero extremal weight module. A uniform model of extremal level-zero crystals using Nakajima monomials was given in [HN06], but the projection onto B was done in type-by-type. The connection of (resp. Demazure) characters of B with (resp. non-symmetric) Macdonald polynomials was given in [LNS⁺15, LNS⁺16b] (resp. [LNS⁺17]).

KR crystals also have a number of other additional properties. Their characters (resp. q -characters) give solutions to Q-systems (resp. T-systems) [Nak01, Nak03, Nak10, Her10]. The existence and combinatorial structure of $B^{r,s}$ was given for non-exceptional types in [FOS09, Oka07, OS08] and a few other special cases [JS10, KMOY07, Yam98]. KR crystals are conjectured to generally be perfect, which is known for non-exceptional types [FOS10] and some other special cases [KMOY07, Yam98].

While many special cases of the conjectured bijection Φ are known (as mentioned above), the description of Φ is given in a type-by-type fashion, meaning that there is no natural extension to the other exceptional types. The original goal of this paper was to extend Φ to $\bigotimes_{i=1}^N B^{r_i,1}$ for type $E_{6,7,8}^{(1)}$. However, it soon became apparent that our description of Φ could be given uniformly for dual untwisted types, and moreover, the proofs given here are uniform.

Explicitly, the core of our main result is a description of Φ when the basic map δ removes the left-most factor $B^{r,1}$, where $\overline{\Lambda}_r$ is either a minuscule weight or the highest root¹ (i.e., it is the perfect crystal of [BFKL06] or $B(\overline{\Lambda}_r)$ is the (“little”) adjoint representation). We then extend the bijection to $B = \bigotimes_{i=1}^N B^{r_i,1}$. As stated above, the description and proof of this is uniform for all dual untwisted types. For the remaining types, we give a uniform description using virtual crystals, and while our proof is essentially uniform, it does contain some type-specific arguments. However, the last part of our main results are that we give a uniform proof that the virtualization map commutes with the bijection Φ .

We show, type-by-type, that these descriptions of δ are equivalent to those described in [KSS02, bM12b, OSS03a, Sch05, Scr16]. As a secondary result, we provide further evidence of the conjecture that KR crystals $B^{r,s}$ in the exceptional types correspond crystal bases of KR modules and of the $X = M$ conjecture by describing the fermionic formula agrees with the conjectured decompositions of [HKO⁺99, HKO⁺02b]. We also describe the so-called Kirillov–Reshetikhin (KR) tableaux for $B^{r,1}$ in types $E_{6,7,8}^{(1)}$, $E_6^{(2)}$, $F_4^{(1)}$, and $G_2^{(1)}$. For certain r , we describe the KR tableaux for $B^{r,s}$ and show that Φ gives a bijection for the single tensor factor.

We are further able to extend our bijection for $\bigotimes_{i=1}^N B^{r_i,s_i}$ when $\overline{\Lambda}_{r_i}$ is a minuscule weight by using the tableaux description given in [JS10]. Specifically, the tableaux can be thought of as single rows that are

¹Our results also include type $A_n^{(1)}$, where we instead have $B^{1,1} \otimes B^{n,1}$ as the atomic object.

weakly increasing with entries in $B(\overline{\Lambda}_{r_i})$, which is naturally considered as a poset. Moreover, the proofs that we give are also uniform. This is the generalization of the results of [SS06a].

Our results is evidence that there should be a natural bijection between rigged configurations and the aforementioned models for KR crystals. Additionally, it also suggests that there should be a uniform description of the $U'_q(\mathfrak{g})$ -crystal structure on rigged configurations by considering the Demazure subcrystal of $B(\ell\Lambda_0)$ following [FSS07, ST12]. Furthermore, our results and our proof techniques are further evidence that the map η that replaces riggings with coriggings in our setting of rigged configurations, which is key in the proof that Φ preserves statistics, is connected with the $*$ -involution on $B(\infty)$. Our results also give a uniform description of the combinatorial R -matrix, which a uniform description was given on the alcove path model in [LL15b], but our proof is type-specific.

Our results also gives further evidence that rigged configurations are intimately connected with the Weyl chamber geometry. Indeed, as rigged configurations are well-behaved under virtualization, the results of [PS17] gives the first evidence. Yet, it is the fact that our results are given for the types where the fundamental alcove is translated by precisely α_a (there are some slight modifications needed for type $A_{2n}^{(2)}$) to another alcove is further evidence. This is additionally emphasized with our result showing Φ intertwines with the virtualization map, extending results of [OSS03b, OSS03c, SS15b]. Furthermore, the related results of [OSS03a] for the untwisted non-simply-laced types appears to be related to our descriptions when the rigged partition $(\nu, J)^{(a)}$ is scaled by the coefficient of α_a required to translate the fundamental alcove. Making this explicit would lead to a completely uniform description of Φ and more strongly link rigged configurations to the underlying geometry.

This paper is organized as follows. In Section 2, we describe the necessary background on crystals, KR crystals, rigged configurations, and the bijection Φ . In Section 3, we describe the map δ for minuscule nodes for dual untwisted types. In Section 4, we describe the map δ for the adjoint node for dual untwisted types. In Section 5, we extend the left-box map for dual untwisted types. In Section 6, we show that the map δ for untwisted types is well-defined by using the virtualization map to the corresponding dual type. In Section 7, we give our main results. In Section 8, we show that our description of Φ is the same as the KSS bijections. In Section 9, we describe the highest weight rigged configurations and KR tableaux for $B^{r,s}$ in a number of different cases. In Section 10, we give our concluding remarks.

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2. BACKGROUND

In this section, we provide the necessary background.

2.1. Crystals. Let \mathfrak{g} be an affine Kac–Moody Lie algebra with index set I , Cartan matrix $(A_{ab})_{a,b \in I}$, simple roots $(\alpha_a)_{a \in I}$, simple coroots $(\alpha_a^\vee)_{a \in I}$, fundamental weights $(\Lambda_a)_{a \in I}$, weight lattice $P = \text{span}\{\Lambda_a \mid a \in I\}$, and canonical pairing $\langle \cdot, \cdot \rangle: P^\vee \times P \rightarrow \mathbb{Z}$ given by $\langle \alpha_a^\vee, \alpha_b \rangle = A_{ab}$. We note that we follow the labeling given in [Kac90] (see Table 1 for the exceptional types and their labellings). Let \mathfrak{g}_0 denote the canonical simple Lie algebra given by the index set $I_0 = I \setminus \{0\}$. Let $\overline{\Lambda}_a$ and $\overline{\alpha}_a$ denote the natural projection of Λ_a and α_a , respectively, onto the weight lattice \overline{P} of \mathfrak{g}_0 . Note $(\overline{\alpha}_a)_{a \in I_0}$ are the simple roots in \mathfrak{g}_0 .

Let c_a and c_a^\vee denote the Kac and dual Kac labels [Kac90, Table Aff1-3]. We define

$$t_a := \max\left(\frac{c_a}{c_a^\vee}, c_0^\vee\right), \quad t_a^\vee := \max\left(\frac{c_a^\vee}{c_a}, c_0\right).$$

Let T_a denote the *translation factors*, the smallest factors such that $T_a \alpha_a$ maps the fundamental polygon to another polygon, and note that $T_a = t_a$ except for type $A_{2n}^{(2)}$ (resp. $A_{2n}^{(2)\dagger}$), where we have $T_n = \frac{1}{2}$ (resp. $T_0 = \frac{1}{2}$) and $T_a = 1$ otherwise. The null root is $\delta = \sum_{a \in I} c_a \alpha_a$, and the canonical central element is $c = \sum_{a \in I} c_a^\vee \alpha_a^\vee$. The normalized (symmetric) invariant form $(\cdot | \cdot): P \times P \rightarrow \mathbb{Z}$ is defined by $(\alpha_a | \alpha_b) = \frac{c_a^\vee}{c_a} A_{ab}$. We write $a \sim b$ if $A_{ab} \neq 0$ and $a \neq b$; in other words, the vertices a and b are adjacent in the Dynkin diagram of \mathfrak{g} .

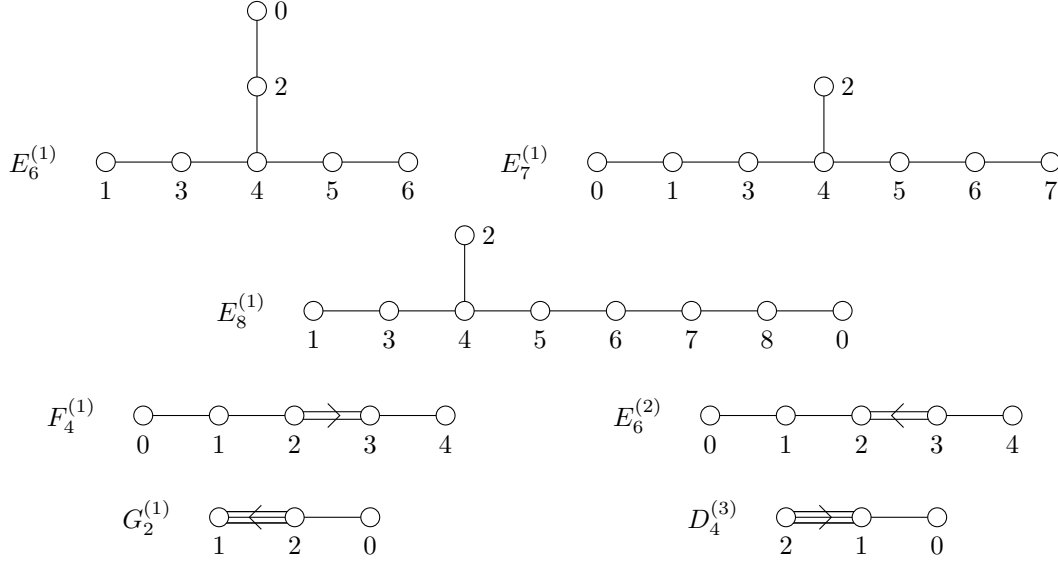


FIGURE 1. Dynkin diagrams for the exceptional affine types.

For \mathfrak{g} not of type $A_n^{(1)}$, let $N_{\mathfrak{g}}$ denote the unique node such that $N_{\mathfrak{g}} \sim 0$, which we call the *adjoint node*. We say a node $r \in I_0$ is *special* if there exists a diagram automorphism $\phi: I \rightarrow I$ such that $\phi(0) = r$. We say a node $r \in I_0$ is *minuscule* if it is special and \mathfrak{g} is of dual twisted affine type.

An *abstract $U_q(\mathfrak{g})$ -crystal* is a set B with the *crystal operators* $e_a, f_a: B \rightarrow B \sqcup \{0\}$, for $a \in I$, and *weight function* $\text{wt}: B \rightarrow P$ that satisfy the following conditions. Let $\varepsilon_a, \varphi_a: B \rightarrow \mathbb{Z}_{\geq 0}$ be statistics given by

$$\varepsilon_a(b) := \max\{k \mid e_a^k b \neq 0\}, \quad \varphi_a(b) := \max\{k \mid f_a^k b \neq 0\}.$$

- (1) $\varphi_a(b) = \varepsilon_a(b) + \langle h_a, \text{wt}(b) \rangle$ for all $b \in B$ and $a \in I$.
- (2) $f_a b = b'$ if and only if $b = e_a b'$ for $b, b' \in B$ and $a \in I$.

The maps e_a and f_a are called the *crystal operators*. We say an element $b \in B$ is *highest weight* if $e_a b = 0$ for all $a \in I$. Define

$$\varepsilon(b) = \sum_{a \in I} \varepsilon_a(b) \Lambda_a, \quad \varphi(b) = \sum_{a \in I} \varphi_a(b) \Lambda_a.$$

Remark 2.1. The abstract crystals we consider in this paper sometimes called *regular* or *seminormal* in the literature.

We call an abstract $U_q(\mathfrak{g})$ -crystal B a *$U_q(\mathfrak{g})$ -crystal* if B is the crystal basis of some $U_q(\mathfrak{g})$ -module. Kashiwara has shown that the irreducible highest weight module $V(\lambda)$ admits a crystal basis [Kas91]. We denote this crystal basis by $B(\lambda)$, and let $u_\lambda \in B(\lambda)$ denote the unique highest weight element and is the unique element of weight λ . In particular, the $U_q(\mathfrak{g}_0)$ -crystal $B(\overline{\Lambda}_r)$ is a minuscule representation, the corresponding finite Weyl group \overline{W} acts transitively on $B(\overline{\Lambda}_r)$, if and only if r is a minuscule node.

We define the *tensor product* of abstract $U_q(\mathfrak{g})$ -crystals B_1 and B_2 as the crystal $B_2 \otimes B_1$ that is the Cartesian product $B_2 \times B_1$ with the crystal structure

$$\begin{aligned} e_a(b_2 \otimes b_1) &= \begin{cases} e_a b_2 \otimes b_1 & \text{if } \varepsilon_a(b_2) > \varphi_a(b_1), \\ b_2 \otimes e_a b_1 & \text{if } \varepsilon_a(b_2) \leq \varphi_a(b_1), \end{cases} \\ f_a(b_2 \otimes b_1) &= \begin{cases} f_a b_2 \otimes b_1 & \text{if } \varepsilon_a(b_2) \geq \varphi_a(b_1), \\ b_2 \otimes f_a b_1 & \text{if } \varepsilon_a(b_2) < \varphi_a(b_1), \end{cases} \\ \varepsilon_a(b_2 \otimes b_1) &= \max(\varepsilon_a(b_1), \varepsilon_a(b_2) - \langle h_a, \text{wt}(b_1) \rangle), \\ \varphi_a(b_2 \otimes b_1) &= \max(\varphi_a(b_2), \varphi_a(b_1) + \langle h_a, \text{wt}(b_2) \rangle), \\ \text{wt}(b_2 \otimes b_1) &= \text{wt}(b_2) + \text{wt}(b_1). \end{aligned}$$

Remark 2.2. Our tensor product convention is opposite of Kashiwara [Kas91].

Let B_1 and B_2 be two abstract $U_q(\mathfrak{g})$ -crystals. A *crystal morphism* $\psi: B_1 \rightarrow B_2$ is a map $B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ with $\psi(0) = 0$ such that the following properties hold for all $b \in B_1$:

- (1) If $\psi(b) \in B_2$, then $\text{wt}(\psi(b)) = \text{wt}(b)$, $\varepsilon_a(\psi(b)) = \varepsilon_a(b)$, and $\varphi_a(\psi(b)) = \varphi_a(b)$.
- (2) We have $\psi(e_a b) = e_a \psi(b)$ if $\psi(e_a b) \neq 0$ and $e_a \psi(b) \neq 0$.
- (3) We have $\psi(f_a b) = f_a \psi(b)$ if $\psi(f_a b) \neq 0$ and $f_a \psi(b) \neq 0$.

An *embedding* and *isomorphism* is a crystal morphism such that the induced map $B_1 \sqcup \{0\} \rightarrow B_2 \sqcup \{0\}$ is an embedding or bijection respectively. A crystal morphism is *strict* if it commutes with all crystal operators.

In type E_n for $n = 6, 7$, we follow [JS10] and label elements $b \in B(\overline{\Lambda}_n)$ (and in $B(\overline{\Lambda}_1)$ for type E_6) by $X \subset \{1, \overline{1}, 2, \overline{2}, \dots, n, \overline{n}\}$, where we have an $a \in X$ (resp. $\overline{a} \in X$) if and only if $\varphi_a(b) = 1$ (resp. $\varepsilon_a(b) = 1$). To ease notation, we write X as a word in the alphabet $\{1, \overline{1}, \dots, n, \overline{n}\}$.

2.2. Kirillov–Reshetikhin crystals. Let $U'_q(\mathfrak{g}) = U_q([\mathfrak{g}, \mathfrak{g}])$ denote the quantum group of the derived subalgebra of \mathfrak{g} . Denote the corresponding weight lattice by $P' = P/\delta$, and therefore, there is a linear dependence on the simple roots in P' . As we will not be considering $U_q(\mathfrak{g})$ -crystals in this paper, we will abuse notation and also denote the $U'_q(\mathfrak{g})$ -weight lattice by P . If B is a $U'_q(\mathfrak{g})$ -crystal, then we say $b \in B$ is *classically highest weight* if $e_a b = 0$ for all $a \in I_0$.

We consider *Kirillov–Reshetikhin (KR) crystals*, which are the conjectural crystal bases to an important class of finite-dimensional irreducible $U'_q(\mathfrak{g})$ -modules known as *Kirillov–Reshetikhin (KR) modules* [HKO⁺99, HKO⁺02b]. We denote a KR module and crystal by $W^{r,s}$ and $B^{r,s}$, respectively, where $r \in I_0$ and $s \in \mathbb{Z}_{>0}$. KR modules are classified by their Drinfel'd polynomials, and $W^{r,s}$ corresponds to the minimal affinization of $B(s\overline{\Lambda}_r)$ [Cha95, Cha01, CP95a, CP95b, CP96a, CP96b, CP98]. In [OS08], it was shown that KR modules in all non-exceptional types admit crystal bases whose combinatorial structure is given in [FOS09]. For the exceptional types, this has been done in a few special cases [JS10, KKM⁺92b, KMOY07, Yam98]. Moreover, a uniform model for $B^{r,1}$ was given using quantum and projected level-zero LS paths [LNS⁺15, LNS⁺16a, LNS⁺16b, NS06, NS08a, NS08b].

We note that there is a unique element $u(B^{r,s}) \in B^{r,s}$, called the *maximal element*, which is characterized by being the unique element of classical weight $s\overline{\Lambda}_r$ (it is also classically highest weight). Furthermore, it is known that tensor products of KR crystals are connected [FSS07, Oka13], and it is known that the maximal vector of $B \otimes B'$ is given by the tensor product of maximal elements $u \otimes u'$. We define the unique $U'_q(\mathfrak{g})$ -crystal morphism $R: B \otimes B' \rightarrow B' \otimes B$, called the *combinatorial R-matrix*, by $R(u \otimes u') = u' \otimes u$, where u and u' are the maximal weight elements of B and B' respectively.

We say a KR crystal $B^{r,1}$ is *minuscule* if r is a minuscule node. We note that $B^{r,s} \cong B(s\overline{\Lambda}_r)$ as $U_q(\mathfrak{g}_0)$ -crystals if r is a special node.

2.3. Virtual crystals. We recall the definition of a virtual crystal from [OSS03b, OSS03c]. Let $\phi: \widehat{\Gamma} \searrow \Gamma$ denote a folding of the Dynkin diagram $\widehat{\Gamma}$ of $\widehat{\mathfrak{g}}$ onto the Dynkin diagram Γ of \mathfrak{g} that arises from the natural

embeddings [JM85]

$$\begin{aligned}
C_n^{(1)}, A_{2n}^{(2)}, A_{2n}^{(2)\dagger}, D_{n+1}^{(2)} &\hookrightarrow A_{2n-1}^{(1)}, \\
B_n^{(1)}, A_{2n-1}^{(2)} &\hookrightarrow D_{n+1}^{(1)}, \\
F_4^{(1)}, E_6^{(2)} &\hookrightarrow E_6^{(1)}, \\
G_2^{(1)}, D_4^{(3)} &\hookrightarrow D_4^{(4)}.
\end{aligned} \tag{2.1}$$

For ease of notation, if X is an object for type \mathfrak{g} , we denote the corresponding object for type $\widehat{\mathfrak{g}}$ by \widehat{X} . By abuse of notation, let $\phi: \widehat{I} \searrow I$ also denote the corresponding map on the index sets. We define the *scaling factors* $\gamma = (\gamma_a)_{a \in I}$ by

$$\gamma_a = \frac{\max_a T_a}{T_a}.$$

Note that if $|\phi^{-1}(a)| = 1$, then $\gamma_a = 1$, and if \mathfrak{g} is of type $D_{n+1}^{(2)}$ or $A_{2n-1}^{(2)}$, then $\gamma_a = 1$ for all $a \in I$.

Furthermore, we have a natural embedding $\Psi: P \rightarrow \widehat{P}$ given by

$$\Lambda_a \mapsto \gamma_a \sum_{b \in \phi^{-1}(a)} \widehat{\Lambda}_b,$$

and note this induces a similar embedding on the root lattice

$$\alpha_a \mapsto \gamma_a \sum_{b \in \phi^{-1}(a)} \widehat{\alpha}_b$$

and $\Psi(\delta) = c_0 \gamma_0 \widehat{\delta}$.

Definition 2.3. Let \widehat{B} be a $U'_q(\widehat{\mathfrak{g}})$ -crystal and $V \subseteq \widehat{B}$. Let ϕ and $(\gamma_a)_{a \in I}$ be the folding and the scaling factors given above. The *virtual crystal operators* (of type \mathfrak{g}) are defined as

$$e_a^v := \prod_{b \in \phi^{-1}(a)} \widehat{e}_b^{\gamma_a}, \quad f_a^v := \prod_{b \in \phi^{-1}(a)} \widehat{f}_b^{\gamma_a}.$$

A *virtual crystal* is the quadruple $(V, \widehat{B}, \phi, (\gamma_a)_{a \in I})$ such that V has an abstract $U'_q(\mathfrak{g})$ -crystal structure defined by

$$\begin{aligned}
e_a &:= e_a^v, & f_a &:= f_a^v, \\
\varepsilon_a &:= \gamma_a^{-1} \widehat{\varepsilon}_a, & \varphi_a &:= \gamma_a^{-1} \widehat{\varphi}_a, \\
\text{wt} &:= \Psi^{-1} \circ \widehat{\text{wt}},
\end{aligned} \tag{2.2}$$

for any $b \in \phi^{-1}(a)$.

When there is no danger of confusion, we simply denote the virtual crystal by V . We say B *virtualizes* in \widehat{B} if there exists a $U'_q(\mathfrak{g})$ -crystal isomorphism $v: B \rightarrow V$ for some virtual crystal, and we call the resulting isomorphism a *virtualization map*.

It is straightforward to see that virtual crystals are closed under direct sums. Moreover, they are closed under tensor products.

Proposition 2.4 ([OSS03b, Prop. 6.4]). *Virtual crystals form a tensor category.*

Furthermore, KR crystals are conjecturally well-behaved under virtualization.

Conjecture 2.5 ([OSS03b, Conj. 3.7]). *There exists a virtualization map from the KR crystal $B^{r,s}$ into*

$$\widehat{B}^{r,s} = \begin{cases} B^{n,s} \otimes B^{n,s} & \text{if } \mathfrak{g} = A_{2n}^{(2)}, A_{2n}^{(2)\dagger} \text{ and } r = n, \\ \bigotimes_{r' \in \phi^{-1}(r)} B^{r', \gamma_r s} & \text{otherwise.} \end{cases}$$

Conjecture 2.5 is known for all non-exceptional types [Oka13, Thm. 5.1] and $B^{r,1}$ for all other types unless $r = 2$ and \mathfrak{g} is of type $F_4^{(1)}$ [SS15b]. A more uniform proof of Conjecture 2.5 for some special cases using projected level-zero LS paths was recently given in [PS17]; in particular, for $B^{r,1}$ when $\gamma_r = 1$.

2.4. Adjoint crystals. We recall the construction of certain level 1 perfect crystals from [BFKL06]. Define $\theta := \delta/c_0 - \alpha_0$, and so

$$\theta = (c_1\alpha_1 + c_2\alpha_2 + \cdots + c_n\alpha_n)/c_0. \quad (2.3)$$

Let $\Delta = \{\overline{\text{wt}}(b) \mid b \in B(\theta)\} \setminus \{0\}$. The $U_q(\mathfrak{g}_0)$ -crystal $B(\theta)$ has vertices

$$\{x_\alpha \mid \alpha \in \Delta\} \sqcup \{y_a \mid a \in I_0, \alpha_a \in \Delta\}$$

and a -arrows

- $x_\alpha \xrightarrow{a} x_\beta$ if and only if $\alpha - \alpha_a = \beta$, or
- $x_{\alpha_a} \xrightarrow{a} y_a \xrightarrow{a} x_{-\alpha_a}$.

The (classical) weight function $\overline{\text{wt}}$ is given by $\overline{\text{wt}}(x_\alpha) = \alpha$ and $\overline{\text{wt}}(y_a) = 0$. Let $\Delta^\pm := \Delta \cap P^\pm$, and we say $\overline{\text{wt}}(b) > 0$ if $\overline{\text{wt}}(b) \in \Delta^+$ and $\overline{\text{wt}}(b) < 0$ if $\overline{\text{wt}}(b) \in \Delta^-$.

Remark 2.6. If \mathfrak{g} is of untwisted type, then $B(\theta)$ is the adjoint representation and Δ is the set of all roots of \mathfrak{g}_0 . For \mathfrak{g} of twisted type, $B(\theta)$ is the “little” adjoint representation of \mathfrak{g}_0 (with highest weight being the highest short root of \mathfrak{g}_0). For type $A_{2n}^{(2)}$, there does not exist an $a \in I_0$ such that $\alpha_a \in \Delta$. For $A_{2n}^{(2)\dagger}$, then $B(\theta) = B(\overline{\Lambda}_1)$ of type B_n .

Let \emptyset be the unique element of $B(0)$. We then define a $U'_q(\mathfrak{g})$ -crystal $B^{\theta,1}$ by the classical decomposition

$$B^{\theta,1} \cong \begin{cases} B(\theta) & \text{if } \mathfrak{g} \text{ is of type } A_{2n}^{(2)\dagger}, \\ B(\theta) \oplus B(0) & \text{otherwise,} \end{cases}$$

and 0-arrows

- $x_\alpha \xrightarrow{0} x_\beta$ if and only if $\alpha + \theta = \beta$ and $\alpha, \beta \neq \pm\theta$, or
- $x_{-\theta} \xrightarrow{0} \emptyset \xrightarrow{0} x_\theta$.

The weight is given by requiring it to be level zero. That is to say, we let

$$\text{wt}(b) = \overline{\text{wt}}(b) + k_0\Lambda_0, \quad (2.4)$$

under the natural lift $\overline{\Lambda}_a \rightarrow \Lambda_a$ for all $a \in I_0$ and k_0 is such that $\langle c, \text{wt}(\nu, J) \rangle = 0$. Thus we have

$$B^{\theta,1} \cong \begin{cases} B^{n,1} \otimes B^{1,1} & \text{if } \mathfrak{g} \text{ is of type } A_n^{(1)}, \\ B^{N_{\mathfrak{g}},2} & \text{if } \mathfrak{g} \text{ is of type } C_n^{(1)}, \\ B^{N_{\mathfrak{g}},1} & \text{otherwise.} \end{cases}$$

Remark 2.7. In the construction of $B^{\theta,1}$, we can consider $\emptyset = y_0$. Thus, for type $A_{2n}^{(2)\dagger}$, as there are not elements x_{α_0} nor $x_{-\alpha_0}$, we do not include y_0 . This reflects the fact that $A_{2n}^{(2)\dagger}$ is isomorphic to $A_{2n}^{(2)}$ after forgetting about the choice of the affine node.

There exists higher level analogs $B^{\theta,s}$, where as classical crystals, we have $B^{\theta,s} \cong \bigoplus_{k=0}^s B(k\theta)$. The $U'_q(\mathfrak{g})$ -crystal structure is currently known for all non-exceptional types [FOS09, KKM94, KKM⁺92b, Kod09, SS06b] and type $D_4^{(3)}$ [KMOY07]. However, the $U'_q(\mathfrak{g})$ -crystal structure is not known in general, much less uniformly. One potential approach might be to generalize the approach of [Kod09] by examining various embeddings of $B^{\theta,s-1}$ into $B^{\theta,s}$, where the difficulty is overcoming that the multiplicity of the weights of $B(k\theta)$ that do not appear in $B((k-1)\theta)$ are not all 1 in general.

2.5. Rigged configurations. For this section, we assume that \mathfrak{g} is not of type $A_{2n}^{(2)}$ or $A_{2n}^{(2)\dagger}$ for simplicity of the exposition. However, the analogous statements with the necessary modifications for these types may be found in [SS15b].

Denote $\mathcal{H}_0 := I_0 \times \mathbb{Z}_{>0}$. Consider a tensor product of KR crystals $B = \bigotimes_{i=1}^N B^{r_i, s_i}$. A *configuration* $\nu = (\nu^{(a)})_{a \in I_0}$ is a sequence of partitions. Let $m_i^{(a)}$ denote the multiplicity of i in $\nu^{(a)}$. Define the *vacancy*

numbers as

$$\begin{aligned} p_i^{(a)}(\nu; B) &= \sum_{j \in \mathbb{Z}_{>0}} L_j^{(a)} \min(i, j) - \frac{1}{t_a^\vee} \sum_{(b,j) \in \mathcal{H}_0} (\bar{\alpha}_a | \bar{\alpha}_b) \min(t_b v_a i, t_a v_b j) m_j^{(b)} \\ &= \sum_{j \in \mathbb{Z}_{>0}} L_j^{(a)} \min(i, j) - \sum_{b \in I_0} \frac{A_{ab}}{\gamma_b} \sum_{j \in \mathbb{Z}_{>0}} \min(\gamma_a i, \gamma_b j) m_j^{(b)}, \end{aligned} \quad (2.5)$$

where $L_s^{(r)}$ equals the number of factors $B^{r,s}$ that occur in B and

$$v_a = \begin{cases} 2 & a = n \text{ and } \mathfrak{g} = C_n^{(1)}, \\ \frac{1}{2} & a = n \text{ and } \mathfrak{g} = B_n^{(1)}, \\ \frac{1}{2} & a = 3, 4 \text{ and } \mathfrak{g} = F_4^{(1)}, \\ \frac{1}{3} & a = 1 \text{ and } \mathfrak{g} = G_2^{(1)}, \\ 1 & \text{otherwise.} \end{cases}$$

When there is no danger of confusion, we will simply write $p_i^{(a)}$. Note that when $t_a = 1$ for all $a \in I_0$, we have

$$-p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} = L_i^{(a)} - \sum_{b \sim a} A_{ab} m_i^{(b)}. \quad (2.6)$$

Moreover, when $m_i^{(a)} = 0$, we have the *convexity inequality*

$$p_{i-1}^{(a)} + p_{i+1}^{(a)} \leq 2p_i^{(a)} \quad (2.7)$$

or equivalently $-p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} \geq 0$.

Remark 2.8. The values $(v_a)_{a \in I_0}$ arise from a different convention for rigged configurations than those used in, e.g., [OSS03a].

A *B-rigged configuration* is the pair (ν, J) , where ν is a configuration and $J = (J_i^{(a)})_{(a,i) \in \mathcal{H}_0}$ be such that $J_i^{(a)}$ is a multiset $\{x \in \mathbb{Z} \mid x \leq p_i^{(a)}\}$ with $|J_i^{(a)}| = m_i^{(a)}$ for all $(a, i) \in \mathcal{H}_0$. When B is clear, we call a B -rigged configuration simply a rigged configuration. A *highest weight* rigged configuration is a rigged configuration (ν, J) such that $\min J_i^{(a)} \geq 0$ for all $(a, i) \in \mathcal{H}_0$ such that $m_i^{(a)} > 0$. Let $\text{RC}^{HW}(B)$ denote the set of all highest weight B -rigged configurations.

The integers in $J_i^{(a)}$ are called *riggings* or *labels*. The *corriging* or *colabel* of a rigging $x \in J_i^{(a)}$ is defined as $p_i^{(a)} - x$. We note that we can associate a row of length $i \in \nu^{(a)}$ with a rigging x , and we call such a pair (i, x) a *string*. We identify each row of the partition $\nu^{(a)}$ with its corresponding string.

Next, let $\text{RC}(B)$ denote the closure of $\text{RC}^{HW}(B)$ under the following crystal operators. Fix a rigged configuration (ν, J) and $a \in I_0$. For simplicity, we assume there exists a row of length 0 in $\nu^{(a)}$ with a rigging of 0. Let $x = \min\{\min J_i^{(a)} \mid i \in \mathbb{Z}_{>0}\}$; i.e., the smallest rigging in $(\nu, J)^{(a)}$.

e_a : If $x = 0$, then define $e_a(\nu, J) = 0$. Otherwise, remove a box from the smallest row with rigging x , replace that rigging with $x + 1$, and change all other riggings so that the corrigings remain fixed. The result is $e_a(\nu, J)$.

f_a : Add a box from the largest row with rigging x , replace that rigging with $x - 1$, and change all other riggings so that the corrigings remain fixed. The result is $f_a(\nu, J)$ unless the result is not a valid rigged configuration, in which case $f_a(\nu, J) = 0$.

We can extend this to a full $U_q(\mathfrak{g}_0)$ -crystal structure on $\text{RC}(B)$ by

$$\overline{\text{wt}}(\nu, J) = \sum_{(a,i) \in \mathcal{H}_0} i \left(L_i^{(a)} \bar{\Lambda}_a - m_i^{(a)} \bar{\alpha}_a \right).$$

We note that

$$\langle \alpha_a^\vee, \overline{\text{wt}}(\nu, J) \rangle = p_\infty^{(a)}, \quad (2.8)$$

and we can extend the classical weight $\overline{\text{wt}}: \text{RC}(B) \rightarrow \bar{P}$ to the affine weight $\text{wt}: \text{RC}(B) \rightarrow P$ as in Equation (2.4).

Theorem 2.9 ([Sch06, SS15b]). Let B be a tensor product of KR crystals. Fix some $(\nu, J) \in \text{RC}^{HW}(B)$. Let $X_{(\nu, J)}$ denote the closure of (ν, J) under e_a and f_a for all $a \in I_0$. Then $X_{(\nu, J)} \cong B(\lambda)$, where $\lambda = \overline{\text{wt}}(\nu, J)$.

Furthermore, we have the following way to compute the statistics ε_a and φ_a on a rigged configuration.

Proposition 2.10 ([Sch06, SS15b]). Let $x = \min\{\min J_i^{(a)} \mid i \in \mathbb{Z}_{>0}\}$; i.e., the smallest rigging in $(\nu, J)^{(a)}$. We have

$$\varepsilon_a(\nu, J) = -x, \quad \varphi_a(\nu, J) = p_\infty^{(a)} - x.$$

Remark 2.11. Proposition 2.10 states that we could define $f_a(\nu, J) = 0$ if and only if $\varphi_a(\nu, J) = p_\infty^{(a)} - x = 0$.

We will need the *complement rigging* involution $\eta: \text{RC}(B) \rightarrow \text{RC}(B^{\text{rev}})$, where B^{rev} is B in the reverse order. The map η is given on highest weight rigged configurations by replacing each rigging with its corresponding corigging and then extended as a $U_q(\mathfrak{g}_0)$ -crystal isomorphism.

Additionally, rigged configurations are known to be well-behaved under virtualization [OSS03b, OSS03c, SS15b]. We construct a virtualization map $v: \text{RC}(B) \rightarrow \text{RC}(\hat{B})$, where the virtual rigged configuration $(\hat{\nu}, \hat{J}) := v(\nu, J)$ is given by

$$\hat{m}_{\gamma_a i}^{(b)} = m_i^{(a)}, \quad (2.9a)$$

$$\hat{J}_{\gamma_a i}^{(b)} = \gamma_a J_i^{(a)}, \quad (2.9b)$$

for all $b \in \phi^{-1}(a)$. Moreover, we have

$$\hat{p}_{\gamma_a i}^{(b)} = \gamma_a p_i^{(a)} \quad (2.10)$$

for all $b \in \phi^{-1}(a)$.

2.6. Statistics. We now describe two important statistics that arise from mathematical physics. The first is defined on tensor products of KR crystals and the second is defined on rigged configurations.

Let $B^{r,s}$ and $B^{r',s'}$ be KR crystals of type \mathfrak{g} . The *local energy function* $H: B^{r,s} \otimes B^{r',s'} \rightarrow \mathbb{Z}$ is defined as follows. Let $\tilde{b}' \otimes \tilde{b} = R(b \otimes b')$, and define the following conditions:

- (LL) $e_0(b \otimes b') = e_0 b \otimes b'$ and $e_0(\tilde{b}' \otimes \tilde{b}) = e_0 \tilde{b}' \otimes \tilde{b}$;
- (RR) $e_0(b \otimes b') = b \otimes e_0 b'$ and $e_0(\tilde{b}' \otimes \tilde{b}) = \tilde{b}' \otimes e_0 \tilde{b}$.

The local energy function is given by

$$H(e_i(b \otimes b')) = H(b \otimes b') + \begin{cases} -1 & \text{if } i = 0 \text{ and (LL),} \\ 1 & \text{if } i = 0 \text{ and (RR),} \\ 0 & \text{otherwise,} \end{cases} \quad (2.11)$$

and it is known H is defined up to an additive constant [KKM⁺92a]. We normalize H by the condition $H(u(B^{r,s}) \otimes u(B^{r',s'})) = 0$.

Next consider $B^{r,s}$, and let b^\sharp be the unique element such that $\varphi(b^\sharp) = \ell \Lambda_0$, where $\ell = \min\{\langle c, \varphi(b) \rangle \mid b \in B^{r,s}\}$. We then define $D_{B^{r,s}}: B^{r,s} \rightarrow \mathbb{Z}$ by

$$D_{B^{r,s}}(b) = H(b \otimes b^\sharp) - H(u(B^{r,s}) \otimes b^\sharp).$$

Let $B = \bigotimes_{i=1}^N B^{r_i, s_i}$. We define *energy* [HKO⁺99] $D: B \rightarrow \mathbb{Z}$ by

$$\begin{aligned} D(b_N \otimes \cdots \otimes b_1) &= \sum_{1 \leq i < j \leq N} H_i R_{i+1} R_{i+2} \cdots R_{j-1} \\ &\quad + \sum_{j=1}^N D_{B^{r_j, s_j}} R_1 R_2 \cdots R_{j-1}, \end{aligned} \quad (2.12)$$

where R_i and H_i are the combinatorial R -matrix and local energy function, respectively, acting on the i -th and $(i+1)$ -th factors and $D_{B^{r_j, s_j}}$ acts on the rightmost factor. Note that D is constant on classical components since H is and R is a $U'_q(\mathfrak{g})$ -crystal isomorphism.

For rigged configurations, we define a statistic called *cocharge* as follows. First consider a configuration ν , and we define the cocharge of ν by

$$\text{cc}(\nu) = \frac{1}{2} \sum_{\substack{(a,i) \in \mathcal{H}_0 \\ (b,j) \in \mathcal{H}_0}} (\bar{\alpha}_a | \bar{\alpha}_b) \min(t_b v_a i, t_a v_b j) m_i^{(a)} m_j^{(b)}. \quad (2.13)$$

To obtain the cocharge of a rigged configuration (ν, J) , we add all of the riggings to $\text{cc}(\nu)$:

$$\text{cc}(\nu, J) = \text{cc}(\nu) + \sum_{(a,i) \in \mathcal{H}_0} t_a^\vee \sum_{x \in J_i^{(a)}} x. \quad (2.14)$$

Moreover, it is known that cocharge is invariant under the classical crystal operators.

Proposition 2.12 ([Sch06, SS15b]). *Fix a classical component $X_{(\nu,J)}$ as given in Theorem 2.9. The cocharge cc is constant on $X_{(\nu,J)}$.*

The *one-dimensional sum* is defined as

$$X(B, \lambda; q) = \sum_{b \in \mathcal{P}(B; \lambda)} q^{D(b)}, \quad (2.15)$$

where $\mathcal{P}(B; \lambda)$ denotes the classically highest weight elements of B of classical weight λ . The *fermionic formula* is defined as

$$M(B, \lambda; q) = \sum_{\nu \in \mathcal{C}(B; \lambda)} q^{\text{cc}(\nu)} \prod_{(a,i) \in \mathcal{H}_0} \left[\begin{matrix} m_i^{(a)} + p_i^{(a)} \\ m_i^{(a)} \end{matrix} \right]_q, \quad (2.16)$$

where $\mathcal{C}(B; \lambda)$ are all B -configurations of classical weight λ . Note that $J_i^{(a)}$ of a highest weight rigged configuration can be considered as a partition in a $p_i^{(a)} \times m_i^{(a)}$ box for all $(a, i) \in \mathcal{H}_0$. Thus we can write

$$M(B, \lambda; q) = \sum_{(\nu, J) \in \text{RC}^{HW}(B; \lambda)} q^{\text{cc}(\nu, J)}.$$

Now we recall the $X = M$ conjecture of [HKO⁺99, HKO⁺02b].²

Conjecture 2.13. *Let B be a tensor product of KR crystals of type \mathfrak{g} . Then we have*

$$X(B, \lambda; q) = M(B, \lambda; q).$$

Consider a virtualization map $v: B \rightarrow B^v$. We first define the virtual combinatorial R -matrix $R^v: B^v \otimes B'^v \rightarrow B'^v \otimes B^v$ as the restriction of the type $\hat{\mathfrak{g}}$ combinatorial R -matrix \hat{R} to the image of v . We note that it is not clear that R^v is well-defined, but this follows for $B^{r,1} \otimes B^{r',1}$ for dual untwisted types from the results of [LNS⁺15, LNS⁺16b, LL15b, PS17]. Thus, we may define virtual analogs of (local) energy and cocharge by

$$\begin{aligned} H^v(b \otimes b') &:= \gamma_0^{-1} \hat{H}(v(b) \otimes v(b')), \\ D^v(b) &:= \gamma_0^{-1} \hat{D}(v(b)), \\ \text{cc}^v(\nu, J) &:= \gamma_0^{-1} \hat{\text{cc}}(\hat{\nu}, \hat{J}). \end{aligned}$$

Hence, we define

$$\begin{aligned} X^v(B, \lambda; q) &= \sum_{b \in \mathcal{P}(B; \lambda)} q^{D^v(b)}, \\ M^v(B, \lambda; q) &= \sum_{(\nu, J) \in \text{RC}^{HW}(B; \lambda)} q^{\text{cc}^v(\nu, J)}. \end{aligned}$$

²To obtain the formulas of [HKO⁺99, HKO⁺02b], we need to substitute $q = q^{-1}$.

Proposition 2.14 ([OSS03c]). *Let B^v be a virtual crystal of B . Then we have*

$$\begin{aligned} D^v(b) &= D(b), \\ \text{cc}^v(\nu, J) &= \text{cc}(\nu, J). \end{aligned}$$

Moreover, we have

$$\begin{aligned} X^v(B, \lambda; q) &= X(B, \lambda; q), \\ M^v(B, \lambda; q) &= M(B, \lambda; q). \end{aligned}$$

2.7. Kleber algorithm. We first recall the *Kleber algorithm* [Kle98] for when \mathfrak{g} is an affine type such that \mathfrak{g}_0 is simply-laced. For $x, y \in \overline{P}^+$, let $d_{xy} := x - y$.

Definition 2.15 (Kleber algorithm). Let B be a tensor product of KR crystals of type \mathfrak{g} . The *Kleber tree* $T(B)$ is a tree whose nodes will be given by weights in \overline{P}^+ and edges are labeled by $d_{xy} \in \overline{Q}^+ \setminus \{0\}$ and constructed recursively as follows. Begin with T_0 being the tree consisting of a single node of weight 0. We then do the following steps starting with $\ell = 1$.

- (K1) Let T'_ℓ be obtained from $T_{\ell-1}$ by adding $\sum_{a=1}^n \overline{\Lambda}_a \sum_{i \geq \ell} L_i^{(a)}$ to the weight of each node.
- (K2) Construct T_ℓ from T'_ℓ as follows. Let x be a node at depth $\ell - 1$. Suppose there is a weight $y \in \overline{P}^+$ such that $d_{xy} \in \overline{Q}^+ \setminus \{0\}$. If x is not the root, then let w be the parent of x . Then we have $d_{wx} - d_{xy} \in \overline{Q}^+ \setminus \{0\}$. For all such y , attach y as a child of x .
- (K3) If $T_\ell \neq T'_{\ell-1}$, then repeat from (K1); otherwise terminate and return $T(B) = T_\ell$.

Now we convert the tree to highest weight rigged configurations as follows. Let x be a node at depth p in $T(B)$, and $x_0, x_1, \dots, x_p = x$ be the weights of nodes on the path from the root of $T(B)$ to x . The resulting configuration ν is given by

$$m_i^{(a)} = (x_{i-1} - 2x_i + x_{i+1} \mid \overline{\Lambda}_a)$$

where we make the convention that $x = x_j$ for all $j > p$. We then take the riggings over all possible values between 0 and $p_i^{(a)}$.

Remark 2.16. We can reformulate the construction of the configuration ν in the following ways. Suppose $d_{x_{i-1}x_i} = \sum_{b \in I} k_i^{(b)} \alpha_b$ for all $1 \leq i \leq p$. There are $k_i^{(a)}$ rows of length i in $\nu^{(a)}$. We also have $\nu^{(a)}$ equal to the transpose of the partition $k_1^{(a)} k_2^{(a)} \cdots k_p^{(a)}$, or we stack a column of height $k_a^{(i)}$ over all i .

When \mathfrak{g}_0 is of non-simply-laced type, we use the *virtual Kleber algorithm* [OSS03c] by using virtual rigged configurations.

Definition 2.17 (Virtual Kleber algorithm). The *virtual Kleber tree* is defined from the Kleber tree of \widehat{B} in the ambient type, but we only add a child y to x in step (K2) if the following conditions are satisfied:

- (V1) $(y \mid \widehat{\alpha}_b) = (y \mid \widehat{\alpha}_{b'})$ for all $b, b' \in \phi^{-1}(a)$.
- (V2) For all $a \in I_0$, if $\ell - 1 \notin \gamma_a \mathbb{Z}$, then the coefficient of $\overline{\alpha}_a$ in d_{wx} and d_{xy} , where w is the parent of x , must be equal.

Let $\widehat{T}(B)$ be the resulting tree, which we will call the *ambient tree*. Let $\gamma = \max_{a \in I} \gamma_a$. We now select nodes which satisfy either:

- (A1) y is at depth $\ell \in \gamma \mathbb{Z}$, or
- (A2) $(d_{xy} \mid \widehat{\Lambda}_a) = 0$ for every a such that $1 < \gamma = \gamma_a$, where x is the parent of y .

We construct the final rigged configurations from the selected nodes by devirtualizing the (virtual) rigged configurations obtained from the usual Kleber algorithm satisfying Equation (2.9b) (note that Equation (2.9a) is satisfied by (V1) and (V2)).

2.8. KSS-type bijection. In this section, we describe the (conjectural) KSS-type bijection $\widetilde{\Phi}: \text{RC}(B) \rightarrow B$.

Let B be a tensor product of KR crystals. We consider B expressed in terms of the so-called *Kirillov–Reshetikhin (KR) tableaux* of [OSS13, SS15b, Scr16]. KR tableaux, generally speaking, are $r \times s$ rectangular

tableaux filled with entries of $B^{1,1}$ and determined by their classically highest weight elements. Following [KSS02], we define a map $\tilde{\Phi}: \text{RC}(B) \rightarrow B$ recursively by the composition of

$$\begin{aligned}
& \tilde{\delta}: B^{1,1} \otimes B^\bullet \rightarrow B^\bullet, \\
& \boxed{b} \otimes b^\bullet \mapsto b^\bullet, \\
& \text{lb}: B^{r,1} \otimes B^\bullet \rightarrow B^{1,1} \otimes B^{r-1,1} \otimes B^\bullet \quad (r \neq 1), \\
& \begin{array}{|c|} \hline b_1 \\ \hline \vdots \\ \hline b_{r-1} \\ \hline b_r \\ \hline \end{array} \otimes b^\bullet \mapsto \boxed{b_r} \otimes \begin{array}{|c|} \hline b_1 \\ \hline \vdots \\ \hline b_{r-1} \\ \hline \end{array} \otimes b^\bullet, \\
& \text{ls}: B^{r,s} \otimes B^\bullet \rightarrow B^{r,1} \otimes B^{r,s-1} \otimes B^\bullet \quad (s \geq 2), \\
& \begin{array}{|c|c|c|c|} \hline b_{11} & b_{12} & \cdots & b_{1s} \\ \hline \vdots & \vdots & \ddots & \vdots \\ \hline b_{r1} & b_{r2} & \cdots & b_{rs} \\ \hline \end{array} \otimes b^\bullet \mapsto \begin{array}{|c|} \hline b_{11} \\ \hline \vdots \\ \hline b_{r1} \\ \hline \end{array} \otimes \begin{array}{|c|c|c|c|} \hline b_{12} & \cdots & b_{1s} \\ \hline \vdots & \ddots & \vdots \\ \hline b_{r2} & \cdots & b_{rs} \\ \hline \end{array} \otimes b^\bullet,
\end{aligned}$$

and their corresponding maps on rigged configurations. We do not explicitly recall the map $\tilde{\delta}$ on rigged configurations here as it strongly depends upon type and we later give a more uniform description of the map. Instead we refer the reader to [OSS03a, OS12, Scr16] for the explicit (type-dependent) descriptions. Note that $\tilde{\delta}$ is currently only described/known for non-exceptional affine types, type $E_6^{(1)}$, and type $D_4^{(3)}$.³ Moreover, $\tilde{\delta}$ is the key component of the bijection. The map lb is given for all non-exceptional types by adding a length 1 singular row to all $\nu^{(a)}$ for all $a < r$. The map ls is the identity map.

We recall and consolidate some of the conjectures given in [SS15b] and has been known to experts prior.

Conjecture 2.18. *Let B be a tensor product of KR crystals of affine type. Then $\tilde{\Phi}: \text{RC}(B) \rightarrow B$ is a (classical) crystal isomorphism such that $D \circ \tilde{\Phi} \circ \eta = \text{cc}$ and the diagram*

$$\begin{array}{ccc}
\text{RC}(B) & \xrightarrow{\tilde{\Phi}} & B \\
\text{id} \downarrow & & \downarrow R \\
\text{RC}(B') & \xrightarrow{\tilde{\Phi}} & B'
\end{array}$$

commutes, where B' is a reordering of the factors of B .

When we restrict $\tilde{\Phi}$ to classically highest weight elements, this gives a combinatorial proof of the $X = M$ conjecture of [HKO⁺99, HKO⁺02b].

In type $A_n^{(1)}$, it was shown in [KSS02] that Conjecture 2.18 holds on classically highest weight elements, and as such, the analogous (conjectural) bijections are known as KSS-type bijections. This was an extension of the pioneering work of Kerov, Kirillov, and Reshetikhin in [KR86, KKR86], where they showed Conjecture 2.18 is true for classically highest weight elements in the special case $B = (B^{1,1})^{\otimes N}$ of type $A_n^{(1)}$. In [DS06], it was shown that $\tilde{\Phi}$ is a classical crystal isomorphism in type $A_n^{(1)}$ and a $U'_q(\mathfrak{g})$ -crystal isomorphism in [SW10]. Furthermore, Conjecture 2.18 is known to (partially) hold in many different special cases for classically highest weight elements across the non-exceptional types [OSS03a, OSS03b, OSS03c, OSS13, Sch05, SS06a, SS15b] and some progress has been made in the exceptional types [OS12, Scr16]. Furthermore, in [Sak14], it was shown that $\tilde{\Phi}$ is a classical crystal isomorphism in type $D_n^{(1)}$. Recently, the general case for type $D_n^{(1)}$ was proven in [OSSS16].

As we will need it later on, we recall the general steps of the proof that $\tilde{\Phi}$ is a statistic preserving bijection on highest weight elements for $B = \bigotimes_{k=1}^N B^{1,1}$ when $r_k = 1$ for all k . Let $(\nu, J) \in \text{RC}^{HW}(B; \lambda)$. Define

³A map for when the left factor is $B^{2,1}$ of type $E_6^{(1)}$ was conjectured in [bM12b].

$(\overline{\nu}, \overline{J}) = \tilde{\delta}(\nu, J)$ and let r be the return value from $\tilde{\delta}$. Denote by $\beta_1^{(r_N)}, \overline{\beta}_1^{(r_N)}$ the length of the first column in $\nu^{(r_N)}, \overline{\nu}^{(r_N)}$, respectively.

There are five things the need to be verified to show that $\tilde{\Phi}$ is a statistic preserving bijection on classically highest weight elements for $B = B^{r_N, 1} \otimes B^\bullet$, where B^\bullet is a tensor product of KR crystals:

- (I) $\lambda - \overline{\text{wt}}(r)$ is dominant.
- (II) $\tilde{\delta}(\nu, J) \in \text{RC}^{HW}(B^\bullet, \lambda - \text{wt}(r))$.
- (III) b can be appended to $(\overline{\nu}, \overline{J})$ to give (ν, J) .
- (IV) For $N \geq 2$, we have

$$\text{cc}(\nu, J) - \text{cc}(\overline{\nu}, \overline{J}) = \frac{t_{r_N}^\vee}{c_0^\vee} \beta_1^{(r_N)} - \chi(b_N = \emptyset). \quad (2.17)$$

- (V) For $N \geq 2$, we have

$$H(b_N \otimes b_{N-1}) = \frac{t_{r_N}^\vee}{c_0^\vee} (\beta_1^{(r_N)} - \overline{\beta}_1^{(r_N)}) - \chi(b_N = \emptyset) + \chi(b_{N-1} = \emptyset). \quad (2.18)$$

We note that Equation (2.17) and Equation (2.18) are those given in [OSS03a, Lemma 5.1].

Next, we will need dual notions of the maps $\tilde{\delta}$, lb, and ls acting on the right, which we denote by $\tilde{\delta}^*$, rb, and rs. First, we recall the definition of *Lusztig's involution* $\star: B(\lambda) \rightarrow B(\lambda)$, the unique $U_q(\mathfrak{g}_0)$ -crystal involution satisfying

$$(e_i b)^* = f_{\xi(i)} b^*, \quad (f_i b)^* = e_{\xi(i)} b^*, \quad \text{wt}(B^*) = w_0 \text{wt}(b), \quad (2.19)$$

where w_0 is the long element of the Weyl group of \mathfrak{g}_0 and $\xi: I_0 \rightarrow I_0$ is defined by $w_0 \overline{\Lambda}_i \mapsto \overline{\Lambda}_{\xi(i)}$ and $w_0 \overline{\alpha}_i = -\overline{\alpha}_{\xi(i)}$. In particular, Lusztig's involution sends the highest weight element to the lowest weight element. We extend Lusztig's involution to an involution $\star: B^{r,s} \rightarrow B^{r,s}$ by defining $\xi(0) = 0$ and satisfying Equation (2.19) and sends the maximal element to the minimal element, the unique element of weight $-\text{wt}(u(B^{r,s}))$. We also can extend Lusztig's involution to tensor products by a natural isomorphism

$$(B_2 \otimes B_1)^* \cong B_1^* \otimes B_2^*$$

given by $(b_2 \otimes b_1)^* = b_1^* \otimes b_2^*$. Then we define

$$\tilde{\delta}^* := \uparrow \circ \star \circ \tilde{\delta} \circ \star, \quad \text{rb} := \star \circ \text{lb} \circ \star, \quad \text{rs} := \star \circ \text{ls} \circ \star,$$

where $\uparrow(b)$ is the classically highest weight corresponding to b . By considering $\diamond := \uparrow \circ \star$, we also have

$$\tilde{\delta}^* = \diamond \circ \tilde{\delta} \circ \diamond, \quad \text{rb} = \diamond \circ \text{lb} \circ \diamond, \quad \text{rs} = \diamond \circ \text{ls} \circ \diamond.$$

3. MINUSCULE δ FOR DUAL UNTWISTED TYPES

In this section, we describe the map δ used to construct Φ for minuscule fundamental weights when \mathfrak{g} is of dual untwisted affine type. More explicitly, we restrict ourselves to simply-laced affine types and types $A_{2n-1}^{(2)}$ and $D_{n+1}^{(2)}$ as types $D_4^{(3)}$ and $E_6^{(2)}$ do not contain any minuscule fundamental weights. We note that for these types, we have $t_a = 1$ for all $a \in I$. In this section, we give a uniform proof that Φ is a statistic preserving bijection for tensor products of minuscule KR crystals.

3.1. Uniform description. We construct the map $\delta_r: B^{r,1} \otimes B^\bullet \rightarrow B^\bullet$, where B^\bullet is a tensor product of KR crystals and $\overline{\Lambda}_r$ is a minuscule weight of type \mathfrak{g}_0 (i.e., r is a special node) as follows. Start at $b = u_{\overline{\Lambda}_r}$, and set $\ell_0 = 1$. Consider step j . From b , let ℓ_j denote a minimal $i_a \geq \ell_{j-1}$ ($a \in I_0$ also varies) such that $f_a b \neq 0$ and $\nu^{(a)}$ has a singular row of length i_a that has not been previously selected. If no such row exists, terminate, set all $\ell_{j'} = \infty$ for $j' \geq j$, and return b . Otherwise select such a row in $\nu^{(a)}$ and repeat the above with $f_a(b)$.

We form the new rigged configuration by removing a box from each row selected by δ_r , making the resulting rows singular, and keeping all other rows the same.

We note that $\delta_1 = \tilde{\delta}$ was described by Okado and Sano [OS12] for type $E_6^{(1)}$. It is also straightforward to see that $\delta_1 = \tilde{\delta}$ in type $A_n^{(1)}$ (given in [KR86, KKR86]) and type $D_n^{(1)}$ and $A_{2n-1}^{(2)}$ (given in [OSS03a]). We collect these results the following theorem.

Theorem 3.1. Let $B = \bigotimes_{i=1}^N B^{1,1}$ be of type $A_n^{(1)}$, $D_n^{(1)}$, $E_6^{(1)}$, or $A_{2n-1}^{(2)}$. The map $\tilde{\Phi}: \text{RC}(B) \rightarrow B$ is a bijection such that $\tilde{\Phi} \circ \eta$ sends cocharge to energy.

We need a few facts about minuscule representations. Let $\overline{W} = \langle s_a \mid a \in I_0 \rangle$ be the Weyl group of \mathfrak{g}_0 with s_a being the simple reflection corresponding to α_a . The cosets $\overline{W}/\overline{W}_{\hat{r}}$, where $\overline{W}_{\hat{r}}$ is the parabolic subgroup generated by $\langle s_i \mid i \in I_0 \setminus \{r\} \rangle$, parameterize the elements $B(\overline{\Lambda}_r)$. Specifically, we have

$$B(\overline{\Lambda}_r) = \{b_w := f_{a_1} \cdots f_{a_\ell} u_{\overline{\Lambda}_r} \mid w = s_{i_1} \cdots s_{i_\ell} \in \overline{W}/\overline{W}_{\hat{r}}\}, \quad (3.1)$$

where the elements w are the minimal length coset representatives. Furthermore, the reduced expressions of w give all paths from b_w to $u_{\overline{\Lambda}_r}$.

Lemma 3.2. Let $\overline{\Lambda}_r$ be a minuscule weight. Then $0 \leq \varepsilon_a(b) + \varphi_a(b) \leq 1$ for all $b \in B(\overline{\Lambda}_r)$ and $a \in I_0$.

Proof. The claim follows immediately from Equation (3.1). \square

The following lemma is the key fact for minuscule nodes, which is a generalization of [OS12, Lemma 2.1].

Lemma 3.3. Let $\lambda \in P^+$. Then λ is a minuscule weight if and only if the crystal graph of $B(\lambda)$ has the following properties:

- (I) Consider a path P in $B(\overline{\Lambda}_r)$ such that the initial and terminal arrows have the same color a . Then either
 - (a) there are exactly two arrows colored by a' and a'' in P such that $a' \sim a$ and $a'' \sim a$, or
 - (b) there is exactly one arrow colored by a' in P such that $A_{aa'} = -2$.
- (II) Consider a length 2 path with colors (a, a') in $B(\overline{\Lambda}_r)$ with $a \not\sim a'$. Then there exists a length 2 path (a', a) with the same initial and terminal vertices in $B(\overline{\Lambda}_r)$.

Proof. We recall that $\lambda \in P^+$ is minuscule implies λ is a fundamental weight. By Equation (3.1), property (I) is given by [Ste01, Prop. 2.3] and (II) was shown in [Pro99] for the simply-laced case and the general case by [Ste01, Prop. 2.1]. \square

Remark 3.4. The conditions (2) and (4) of [OS12, Lemma 2.1], respectively, are consequences of (I) and (II) of Lemma 3.3, respectively, which correspond to (1) and (3) in [OS12], respectively. Thus we have only stated the necessary properties.

One important consequence of Lemma 3.3 is that the result of δ_r does not depend on the choice of a' such that $i_{a'} = \min_a i_a$ at each step. Another is that the proof given in [OS12] holds in this generality except for a fact about the local energy function (i.e., part (V)). Recall that we need to show $H(b_N \otimes b_{N-1})$ is equal to the number of length 1 singular rows selected by δ_r in $\nu^{(r)}$, which implies that Φ preserves statistics. Note that we have already shown that Φ is a bijection.

3.2. Result for minuscule nodes. We will be using the following key fact about how adding $f_a(b)$ under δ_r^{-1} differs from adding b .

Lemma 3.5. Let (ν, J) and (ν', J') be rigged configurations such that $\delta_r(\nu, J) = \delta_r(\nu', J')$ and the return value of $\delta_r(\nu, J)$ is b and $\delta_r(\nu', J')$ is $f_a(b)$ for some a . Let ℓ be the length of the last selected row during $\delta_r(\nu, J)$. Then (ν', J') differs from (ν, J) by:

- an additional box to a row of length $\ell' \geq \ell$ such that the row is singular;
- any singular rows of length at least ℓ' in (ν, J) in $\nu^{(a')}$, for any a' such that $f_{a'}(f_a b) \neq 0$ and $f_{a'}(b) = 0$, are non-singular in (ν', J') .

Proof. This is straightforward from the definition of δ . \square

Next, we compute the (local) energy function on classically highest weight elements in $B^{r,1} \otimes B^{r,1}$.

Theorem 3.6. Let $\overline{\Lambda}_r$ be a minuscule weight. Then for classically highest weight elements $b \otimes u_{\overline{\Lambda}_r} \in B^{r,1} \otimes B^{r,1}$, we have $H(b \otimes u_{\overline{\Lambda}_r})$ equal to the number of r -arrows in the path from b to $u_{\overline{\Lambda}_r}$ in $B(\overline{\Lambda}_r)$.

Proof. First recall that for $\bar{\Lambda}_r$ to be a minuscule weight, there exists a diagram automorphism ϕ such that $r = \phi(0)$. Therefore, if we consider the finite-type \mathfrak{g}_r given by $I_r := I \setminus \{r\}$, then the corresponding fundamental weight $\check{\Lambda}_0$ is minuscule. We note that $B^{r,1} \cong B(\check{\Lambda}_0)$ as $U_q(\mathfrak{g}_r)$ -crystals, and the classically highest weight element in $B^{r,1}$ is the I_r -lowest weight element in $B(\check{\Lambda}_0)$. Hence, for every classically highest weight element $b \otimes u_{\bar{\Lambda}_r} \in B^{r,1} \otimes B^{r,1}$, there exists a path to $u_{\bar{\Lambda}_r} \otimes u_{\bar{\Lambda}_r}$ using the crystal operators f_a , for $a \in I \setminus \{r\}$. Moreover, the crystal operators only act on the left-most factor since $\varphi_a(u_{\bar{\Lambda}_r}) = \Lambda_r$. The number of 0-arrows is equal to the number of r -arrows in the path from b to $u_{\bar{\Lambda}_r}$ in $B(\bar{\Lambda}_r)$ because $f_r f_0 b = f_0 f_r b$ as $r \not\sim 0$. Hence, we have $H(b \otimes u_{\bar{\Lambda}_r})$ as claimed. \square

It remains to show the local energy function satisfies Equation (2.18).

Lemma 3.7. *Part (V) holds for*

$$B = \bigotimes_{i=1}^N B^{r,1}$$

when $\bar{\Lambda}_r$ is a minuscule weight.

Proof. Note that in order for the second application of δ_r to return $u_{\bar{\Lambda}_r}$, there must not exist any singular rows in $\nu^{(r)}$ after the first application of δ_r . Hence all rows selected in $\nu^{(r)}$ must have length 1. Thus the claim holds on classically highest weight elements.

Thus, to show this holds in the general case of $b' \otimes b$, we use induction on the classical components in $B^{r,1} \otimes B^{r,1}$. We note that we are not applying f_r to the crystal/rigged configuration, but instead looking at how the two left-most factors differ, which results in a box being added to a row in $\nu^{(r)}$.

For all $a \neq r$, the claim follows for $f_a(b' \otimes b)$ since the resulting rigged configuration will differ by an additional box in $\nu^{(a)}$. Thus, it remains to show that the additional box removed to obtain $f_r(b' \otimes b)$ must not have come from a length 1 row as $H(b' \otimes b) = H(f_r(b' \otimes b))$. We consider how the elements change under f_r and split it into two cases.

Suppose $f_r(b' \otimes b) = b' \otimes (f_r b)$. We note that the element b' is unchanged. Hence, the claim follows by induction since it only depends on the first application of δ_r .

Instead suppose $f_r(b' \otimes b) = (f_r b') \otimes b$. Let $(\bar{\nu}, \bar{J}) = \delta_r(\nu, J) = \delta_r(\tilde{\nu}, \tilde{J})$ with a return value of $b' \otimes b$ and $(f_r b') \otimes b$ respectively. Thus, we have $\varepsilon_r(b') \geq \varphi_r(b)$, and we must have $\varepsilon_r(b') = \varphi_r(b) = 0$ because $f_r(b') \neq 0$ and Lemma 3.2. This implies that $b \neq u_{\bar{\Lambda}_r}$ as $\varphi_r(u_{\bar{\Lambda}_r}) = 1$, and so there must exist a singular string in $\bar{\nu}^{(r)}$. Suppose we select only length 1 singular strings in $\tilde{\nu}^{(r)}$, then in order for these to be singular strings in $\bar{\nu}^{(r)}$, we must have followed twice as many a -arrows, for $a \sim r$, than r -arrows. However, by Lemma 3.3, we can only select the same number of a -arrows as r -arrows. This is a contradiction, and hence, the claim follows. \square

Thus, we collect all of our results for this section in the following.

Lemma 3.8. *Let $B = \bigotimes_{i=1}^N B^{r,1}$, where r is a minuscule node. The map*

$$\Phi_r: \text{RC}(B) \rightarrow B,$$

which is given by applying δ_r , is a bijection on classically highest weight elements such that $\Phi_r \circ \eta$ sends cocharge to energy.

4. ADJOINT δ FOR DUAL UNTWISTED TYPES

In this section, we describe the map δ_θ for the adjoint node for dual untwisted types (i.e., $t_a = 1$ for all $a \in I$ or equivalently, \mathfrak{g} is of simply-laced affine type, $A_{2n-1}^{(2)}$, $D_{n+1}^{(2)}$, $D_4^{(3)}$, $E_6^{(2)}$). Furthermore, we give a uniform proof that Φ is a statistic preserving bijection.

4.1. Description of δ_θ . We define the map $\delta_\theta: B^{\theta,1} \otimes B^\bullet \rightarrow B^\bullet$, where B^\bullet is a tensor product of KR crystals, by the following algorithm. Begin with b being the highest weight element in $B(\theta) \subseteq B^{\theta,1}$, and set $\ell_0 = 1$. Consider step j such that $b = x_\beta$, where $\beta > 0$ and $\beta \neq \alpha_a$ for all $a \in I_0$. From b , consider any outgoing arrow labeled by a and find the minimal $i_a \geq \ell_{j-1}$ such that $\nu^{(a)}$ has a singular row of length i_a which has not been previously selected. If no such row exists, terminate, set all $\ell_{j'} = \infty$ for $j' \geq j$ and

$\bar{\ell}_{j'} = \infty$ for all j' , and return b . Otherwise select such a row, set $\ell_j = \min_a i_a$, and repeat the above with $f_{a'}(b)$, where a' is such that $i_{a'} = \min_a i_a$. If $b = x_{\alpha_a}$ for some $a \in I_0$, we do one of the following cases.

- (S) If there exists a singular row of length $i_a \geq \max \ell_{j-1}, 2$, select such a row and set $\ell_j = i_a$.
- (E) If there exists a singular row of length 1 and $\ell_{j-1} = 1$, we terminate, set $\ell_j = 1$ and $\bar{\ell}_{j'} = \infty$ for all j' , and return \emptyset .
- (Q) If there exists a quasisingular row of length $i_a \geq \ell_{j-1}$, we select the quasisingular string and set $\ell_j = i_a$.
- (T) Otherwise we terminate, set $\ell_j = \bar{\ell}_{j'} = \infty$ for all j' , and return x_{α_a} .

If the process has not terminated, set $b = y_a$ and perform the following. Let $\bar{\ell}_0 = \ell_h$, where $h = \sum_{a \in I_0} c_a$, i.e., the height of θ or the number of steps we currently have done. Consider step j , and consider any outgoing arrow labeled by a from b . Find the minimal $i_a \geq \bar{\ell}_{j-1}$ such that $\nu^{(a)}$ has a singular row of length i_a such that

- (D) it had been selected at step j' with $\ell_{j'} = i_a$ or
- (N) it had not been previously selected.

If no such row exists, terminate, set all $\bar{\ell}_{j'} = \infty$ for $j' \geq j$, and return b . Otherwise select such a row, set $\ell_j = \min_a i_a$, redefine $\ell_{j'} := \ell_{j'} - 1$ if case (D) had occurred, and repeat the above with $f_{a'}(b)$, where a' is such that $i_{a'} = \min_a i_a$.

We form the new rigged configuration by

- (1) removing a box from each row each time it was selected by δ (i.e., if Case (D) occurred, then we remove 2 boxes);
- (2) making the resulting rows singular unless Case (Q) occurred, then we make the row selected by $\bar{\ell}_1$ (if $\bar{\ell}_1 \neq \infty$) quasisingular; and
- (3) keeping all other rows the same.

Remark 4.1. This is the (conjectural) map δ of bin Mohammad [bM12b] for \mathfrak{g} of type $E_6^{(1)}$. Moreover, this was the map $\tilde{\delta}$ for \mathfrak{g} of type $D_{n+1}^{(2)}$ in [OSS03a] and of type $D_4^{(3)}$ in [Scr16].

We have the following classification of elements in $B(\theta)$ and will be used to describe the KR tableaux of type $E_8^{(1)}$ and $E_6^{(2)}$.

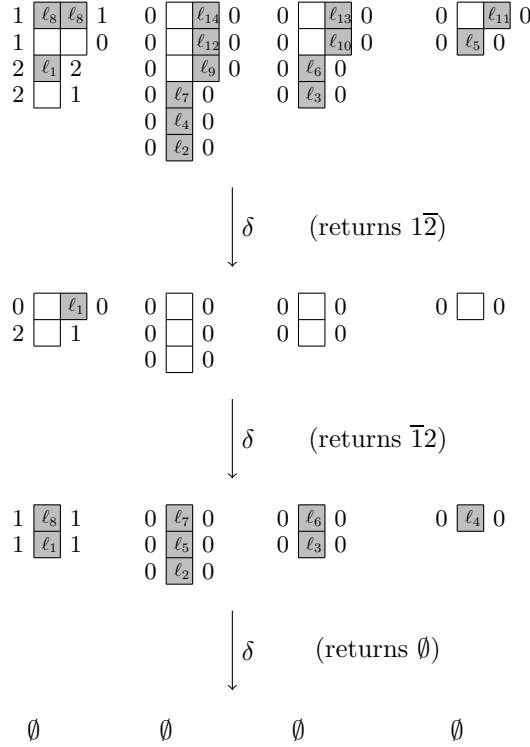
Proposition 4.2. *Let \mathfrak{g} be of simply-laced or twisted type, and fix some $b \in B(\theta)$. Then b has the following properties.*

- b is uniquely determined by ε and φ .
- $\text{wt}(b) = 0$ if and only if there exists a unique $i \in I_0$ such that $\varepsilon_i(b) = \varphi_i(b) = 1$ and $\varepsilon_j(b) = \varphi_j(b) = 0$ for all $j \neq i$.
- $\varepsilon_i(b) = 2$ implies $\varepsilon_j(b) = 0$ for all $j \neq i$.
- $\varphi_i(b) = 2$ implies $\varphi_j(b) = 0$ for all $j \neq i$.

Proof. This follows from the description of $B(\theta)$. □

Thus, similar to types $E_{6,7}$, we can equate our elements in $B(\theta)$ by multisets of $\{1, \bar{1}, 2, \bar{2}, \dots, n, \bar{n}\}$, which as above, we write as words.

Example 4.3. Consider type $E_6^{(2)}$ and $B = (B^{1,1})^{\otimes 4}$. We compute the bijection



and the final application of δ returns 1. Hence, the result of Φ is the element

$$1\bar{2} \otimes \bar{1}2 \otimes \emptyset \otimes 1.$$

As for minuscule representations, there is a bijection between paths x_α to x_θ and reduced expressions of minimal length coset representatives in $\overline{W}/\overline{W}_{\widehat{N}_g}$ (or $\overline{W}/\overline{W}_{\widehat{1},\widehat{n}}$ for type $A_n^{(1)}$). We also have an analog of Lemma 3.3 for $B(\theta)$.

Lemma 4.4. *Consider the path*

$$x_{\alpha_a} \xrightarrow{a} y_a \xrightarrow{a} x_{-\alpha_a}$$

as a single edge. Then (I) and (II) of Lemma 3.3 holds for $B(\theta)$.

Proof. This follows from Proposition 4.2 and that \overline{W} acts transitively on Δ . \square

4.2. Result for adjoint nodes. Now we prove our main result of this section.

Lemma 4.5. *Let \mathfrak{g} be such that $t_a = 1$ for all $a \in I$. Let $B = \bigotimes_{i=1}^N B^{\theta,1}$ be a tensor product of KR crystals. Then the map*

$$\Phi: \text{RC}(B) \rightarrow B$$

is a bijection on classically highest weight elements such that $\Phi \circ \eta$ sends cocharge to energy.

We follow the proof of the KSS-type bijection [OSS03a]. Recall that $p_\infty^{(a)} = \langle h_a, \text{wt}(\nu, J) \rangle$. We give our proof when \mathfrak{g} not of type $A_n^{(1)}$ for simplicity, but the proof for type $A_n^{(1)}$ follows by considering $N_{\mathfrak{g}} = \{1, n\}$.

Proof of (I). Suppose $\tilde{\lambda} = \lambda - \text{wt}(r)$ is not dominant. Thus $r \neq y_a, \emptyset$ as $\text{wt}(y_a) = \text{wt}(\emptyset) = 0$ for all $a \in I$. Thus the only possibilities which make $\tilde{\lambda}$ not dominant for some a is when $0 \leq \langle h_a, \lambda \rangle < \varphi_a(r)$. Note that δ_θ terminates at $\nu^{(a)}$ in each of these cases, and let P_δ denote the path taken by δ_θ . Let $\ell = \nu_1^{(a)}$, i.e., the largest part of $\nu^{(a)}$. Let u_θ denote the highest weight element in $B(\theta) \subseteq B^{N_{\mathfrak{g}},1}$.

We start by assuming $\ell = 0$. Then we have

$$\langle h_a, \lambda \rangle = \sum_{j \in \mathbb{Z}_{>0}} j L_j^{(a)} - \sum_{b \in I_0 \setminus \{a\}} A_{ab} |\nu^{(b)}|. \quad (4.1)$$

Consider the case when $\langle h_a, \lambda \rangle = 0$. If $a = N_{\mathfrak{g}}$, then this is a contradiction since $L_1^{(N_{\mathfrak{g}})} > 0$ and $A_{ab} \leq 0$ for all $b \in I_0 \setminus \{a\}$. Now if $a \neq N_{\mathfrak{g}}$, then $r \neq u_{\theta}$. Thus we must have removed a box from $\nu^{(b)}$ for some $b \sim a$ when performing δ_{θ} . So $-A_{ab} |\nu^{(b)}| > 0$, which is a contradiction. Next, consider the case $\langle h_a, \lambda \rangle > 0$. Hence, we have $\varphi_a(r) > 1$, and so $r \neq u_{\theta}$ (specifically, $r = x_{\alpha_a}$ in the types we consider). Note that either

- there exists a $a' \sim a$ such that δ_{θ} removes a box from $\nu^{(a')}$ with $-A_{aa'} \geq \varphi_a(r)$, or
- there exists a', a'' such that $a', a'' \sim a$ such that δ_{θ} removes a box from $\nu^{(a')}$ and $\nu^{(a'')}$ (if $a' = a''$, then two boxes are removed)

from the crystal structure of $B(\theta)$. (Note this is essentially Lemma 4.4.) Thus, Equation (4.1) implies $\langle h_a, \lambda \rangle \geq \varphi_a(b)$, which is a contradiction.

Now assume that $\ell > 0$. By the definition of the vacancy numbers, we have

$$0 \leq p_i^{(a)} + \sum_{b \in I_0} A_{ab} m_i^{(b)} = \langle h_a, \lambda \rangle.$$

In particular, we have

$$-p_{i-1}^{(a)} + 2p_i^{(a)} - p_{i+1}^{(a)} = L_i^{(a)} - \sum_{b \in I_0} A_{ab} m_i^{(b)}. \quad (4.2)$$

$\langle h_a, \lambda \rangle = 0$:

We note that this is the same proof of (I) for $\ell > 0$ as given in [OS12].

In this case, $\nu^{(a)}$ has a singular string of length ℓ since $0 \geq p_{\ell}^{(a)} \geq J_{\ell}^{(a)} \geq 0$ by convexity. Moreover, we have $m_i^{(b)} = 0$ for all $b \sim a$ and $i > \ell$. If not all rows of length ℓ in $\nu^{(a)}$ have been selected (or doubly selected in Case (S) if we return x_{α} with $\alpha < 0$), then we have a contradiction as we can select a row from m_{ℓ} . Next, by considering the smallest subpath P of P_{δ} ending at r and starting with an a arrow, the fact selecting every row of length ℓ implies

$$2m_{\ell}^{(a)} \leq \sum_{b \sim a} A_{ab} m_i^{(a)} \quad (4.3)$$

by Lemma 4.4. Thus, Equation (4.2) implies that $p_{\ell-1}^{(a)} = 0$ and that Equation (4.3) is an equality. Therefore all rows of length ℓ in $\nu^{(b)}$ for $b \sim a$ are selected by δ between. Hence, when δ_{θ} selected the first row of length ℓ , then all rows of length at most $\ell - 1$ must have been selected by the definition of δ_{θ} , otherwise we would have selected a row of length at most $\ell - 1$ in $\mu^{(b)}$ for $b \sim a$. Thus, we can proceed by induction and show that we select every row of $\nu^{(a)}$.

Therefore, we now have

$$0 = L_1^{(a)} - \sum_{b \in I_0} A_{ab} m_1^{(b)}.$$

If $a = N_{\mathfrak{g}}$, then this is a contradiction from Equation (4.3) and $L_1 \geq 1$. If $a \neq N_{\mathfrak{g}}$, then we must have selected all boxes in length 1 rows of $\nu^{(b)}$ for $b \sim a$ between the first box of a length 1 row of $\nu^{(a)}$. However, to get to a , we must have selected a length 1 row of $\nu^{(b)}$ for some $b \sim a$. This is a contradiction.

$\langle h_a, \lambda \rangle = 1$:

As above, we have $\varphi_a(r) > 1$ and $r = x_{\alpha_a}$. By convexity, we have $p_{\ell}^{(a)} \leq p_{\ell+1}^{(a)} \leq \langle h_a, \lambda \rangle$. If $p_{\ell}^{(a)} = 1$, then we have $m_i^{(a')} = 0$ for all $a' \sim a$ and $i > \ell$ because $\langle h_a, \lambda \rangle = p_i^{(a)} = p_{\ell}^{(a)}$ and Equation (4.2). Furthermore, every row of $\nu^{(a)}$ of length ℓ must be selected and singular, as otherwise δ_{θ} would select such a (quasi)singular row. Therefore, the previous argument holds and results in a contradiction. Hence, we can assume $p_{\ell}^{(a)} = 0$.

Since $0 = p_{\ell}^{(a)} < p_{\ell+1}^{(a)} = 1$, we must have $m_{\ell+1}^{(a')} = 1$ for precisely one $a' \sim a$ with $A_{aa'} = -1$ by Equation (4.2). Note that there exists a singular row of length ℓ in $\nu^{(a)}$. If δ_{θ} can select a row of length ℓ from $\nu^{(a)}$ (whether there is a selectable row or not), then we have a contradiction as above. Therefore, we

must have δ_θ selecting the row of length $\ell + 1$ in $\nu^{(a')}$. If the row of length $\ell + 1$ was the first row of length $\ell + 1$ selected, then we would have chosen this row in $\nu^{(a)}$ before the row in $\nu^{(a')}$, which is a contradiction. Thus suppose δ_θ previously selected a row of length $\ell + 1$ in $\nu^{(a')}$ corresponding to some node $x_\alpha \in B(\theta)$. If $a'' \sim a$, then we have $p_{\ell+1}^{(a)} > 1$, which is a contradiction. We note that $f_a x_\alpha \neq 0$ as the coefficient of $\alpha_{a'}$ in α is still 1. Hence, by Lemma 4.4, the map δ_θ would remove a box from $\nu^{(a)}$ before the one from $\nu^{(a')}$. This is a contradiction. \square

Proof of (II). We need to show that

$$0 \leq \max J_i^{(a)} \leq p_i^{(a)} \quad (4.4)$$

for all $(a, i) \in \mathcal{H}_0$. Considering the algorithm for δ and considering the change in vacancy numbers, the only way to violate Equation (4.4) is when the following cases occur.

- (1) There exists a singular or quasisingular string of length i in $\nu^{(a)}$ such that $\tilde{p}_i^{(a)} - p_i^{(a)} = -1, -2$ respectively and $\ell' \leq i < \ell$.
- (2) We have $m_{\ell-1}^{(a)} = 0$, $p_{\ell-1}^{(a)} = 0$, and $\ell' < \ell$.

In both cases, let ℓ be the length of the row selected by δ in $\nu^{(a)}$ and ℓ' be the length of the row selected immediately before ℓ in $\nu^{(c)}$ such that $c \sim a$.

We will first show that (1) cannot occur. Assume a singular string occurs, then δ would have selected the string of length i , which is a contradiction. Now suppose a quasisingular string occurs which corresponds to $\tilde{p}_i^{(a)} - p_i^{(a)} = -2$, in this case we have $a = 1$ and again, the map δ would have selected this quasisingular string, which is a contradiction.

Now we will show that (2) cannot occur. Let $t < \ell$ be the largest integer such that $m_t^{(a)} > 0$, and if no such t exists, set $t = 0$. By the convexity condition, we have $p_i^{(a)} = p_{\ell-1}^{(a)}$ for all $t < i < \ell$. Thus $m_i^{(c)} = 0$ for all $c \sim a$ and $t \leq i \leq \ell$. Thus $\ell' < t$. If $t = 0$, then this is a contradiction since $1 \leq \ell'$. If $t > 0$, then we have $p_t^{(a)} = 0$ and the row must be singular because $m_t^{(a)} > 0$. Thus we must have $t = \ell$, which is a contradiction. \square

Proof of (III). From the change in vacancy numbers, any string of length not of a length created from δ_θ becomes non-singular. Therefore it is easy to see that the procedure outlined for δ_θ^{-1} is the inverse of δ_θ . \square

Proof of (IV). Recall that $(\overline{\nu}, \overline{J}) = \delta_\theta(\nu, J)$. We can rewrite Equation (2.13) using Equation (2.5) as

$$\text{cc}(\nu) = \frac{1}{2} \sum_{(a,i) \in \mathcal{H}_0} t_a^\vee \left(\sum_{j \in \mathbb{Z}_{>0}} L_j^{(a)} \min(i, j) - p_i^{(a)} \right) m_i^{(a)} \quad (4.5)$$

Next we express $(\ell_k)_k$ as $(\ell_m^{(a_k)})_k$ by the m -th selection in $\nu^{(a_k)}$ by δ . Let $\overline{m}_i^{(a)}$, $\overline{p}_i^{(a)}$, $\overline{J}_i^{(a)}$, and $\overline{L}_i^{(a)}$ denote $m_i^{(a)}$, $p_i^{(a)}$, $J_i^{(a)}$, and $L_i^{(a)}$, respectively, after applying δ , and we have

$$\overline{m}_i^{(a)} = m_i^{(a)} + \sum_m \delta_{\ell_m^{(a)}-1, i} - \delta_{\ell_m^{(a)}, i}, \quad (4.6a)$$

$$\overline{L}_i^{(a)} = L_i^{(a)} - \delta_{a, N_{\mathfrak{g}}} \delta_{i, 1}, \quad (4.6b)$$

Let $\Delta(\text{cc}(\nu, J)) := \text{cc}(\nu, J) - \text{cc}(\overline{\nu}, \overline{J})$. Next, from a straight-forward, but somewhat tedious, calculation using the changes in vacancy numbers, we obtain

$$\begin{aligned} \Delta(\text{cc}(\nu)) &= \sum_{(a,i) \in \mathcal{H}_0} t_a^\vee \left(p_i^{(a)} - \overline{p}_i^{(a)} \right) \left(m_i^{(a)} - \sum_m \delta_{\ell_m^{(a)}, i} \right) - \chi(r = y_a) \\ &\quad + \frac{t_{N_{\mathfrak{g}}}^\vee}{c_0^\vee} \sum_{i \in \mathbb{Z}_{\geq 0}} m_i^{(N_{\mathfrak{g}})} - \chi(r = \emptyset), \end{aligned}$$

$b' \otimes b$	$x_\theta \otimes x_\theta$	$x_{\theta - \alpha_{N_{\mathfrak{g}}}} \otimes x_\theta$	$\emptyset \otimes x_\theta$	$x_\theta \otimes \emptyset$	$x_\alpha \otimes x_\theta$	$y_a \otimes x_\theta$	$x_{-\theta} \otimes x_\theta$	$\emptyset \otimes \emptyset$
$H(b' \otimes b)$	0	1	1	1	2	2	2	2

TABLE 1. Local energy on $B^{\theta,1} \otimes B^{\theta,1}$, where $\alpha \in \Delta^+ \setminus \{\theta - \alpha_{N_{\mathfrak{g}}}\}$ is such that $\alpha + \alpha_{N_{\mathfrak{g}}} \in \Delta^+$.

where r is the return value of δ_θ . Moreover, from the description of δ_θ , we have

$$\Delta(|J|) = \sum_{(a,i) \in \mathcal{H}_0} t_a^\vee (p_i^{(a)} - \bar{p}_i^{(a)}) \left(m_i^{(a)} - \sum_m \delta_{\ell_m^{(a)}, i} \right) + \chi(r = y_a),$$

where the last term is because a quasisingular string was changed into a singular string if that holds. Combining this with the fact that $b_N = r$, we have

$$\Delta(\text{cc}(\nu, J)) = \Delta(\text{cc}(\nu)) + \Delta(|J|) = \frac{t_{N_{\mathfrak{g}}}^\vee}{c_0^\vee} \beta_1^{(N_{\mathfrak{g}})} - \chi(b_N = \emptyset)$$

as desired. \square

Proof of (V). We recall the local energy function on $B^{\theta,1} \otimes B^{\theta,1}$ from [BFKL06] in Table 1, renormalized to our convention. We note that there is a minor error in [BFKL06], where it is stated $H(x_\alpha \otimes x_\theta) = 1$.

Note that in order for the second application of δ_θ to return x_θ or \emptyset , we must not have any singular rows in $\nu^{(N_{\mathfrak{g}})}$. Hence, $\beta_1^{(N_{\mathfrak{g}})} - \bar{\beta}_1^{(N_{\mathfrak{g}})}$ is the number of $N_{\mathfrak{g}}$ -arrows in the path from x_θ to b' from the definition of δ_θ . We note that for every type, we have $c_{N_{\mathfrak{g}}} = 2$, and hence, it is straightforward to see the claim holds in this case.

When $b' \otimes b$ is not classically highest weight, this is similar to the non-highest weight case in the proof of Lemma 3.7 as Lemma 3.5 holds with minor modifications for when the return value is y_a . \square

4.3. Extension to types $C_n^{(1)}$, $A_{2n}^{(2)}$, and $A_{2n}^{(2)\dagger}$. We can also extend our proof to types $C_n^{(1)}$, $A_{2n}^{(2)}$, and $A_{2n}^{(2)\dagger}$.

Since $B^{\theta,1}$ for type $A_{2n}^{(2)}$ (resp., type $A_{2n}^{(2)\dagger}$) does not contain any elements y_a , for $a \in I_0$ (resp., $a = 0$), as noted in Remark 2.6 (resp. Remark 2.7), we modify the definition of δ_θ by removing Case (Q) (resp., Case (E)) as a possibility. Likewise for type $C_n^{(1)}$, we do not have y_a for all $a \in I$, so we modify the definition of δ_θ by removing both Case (Q) and Case (E), but we also need to consider the parts of $\nu^{(n)}$ doubled as per Remark 2.8.

We note that for the proofs above to hold for type $A_{2n}^{(2)}$ and $A_{2n}^{(2)\dagger}$. We have to use the (modified) scaling factors of $\tilde{\gamma}_a = 1$ for all $a \in I$ and the matrix

$$(\tilde{A}_{ab})_{a,b \in I_0} = \begin{bmatrix} 2 & -1 & & & \\ -1 & 2 & -1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 2 & -1 \\ & & & -1 & 1 \end{bmatrix}$$

instead of the Cartan matrix in, e.g., Equation (4.2) and Equation (4.3).

5. EXTENDING THE LEFT-BOX MAP

In this section, we describe a generalization of the left-box map in order to give a tableau description of the crystals $B^{r,1}$ for dual untwisted types. To do so, we first construct **lb-diagrams**, which are digraphs on I_0 such that

- every node has at most one outgoing edge,
- there is a unique sink k , and
- each arrow $r \xrightarrow{b} r'$ is labeled by $b \in B(\bar{\Lambda}_k)$ such that $\varepsilon_a(b) = \delta_{ar'}$ and $\varphi_a(b) = \delta_{ar}$.

For a fixed lb-diagram D , we define the *left-box* on rigged configurations $\text{lb}: \text{RC}(B^{r,1}) \rightarrow \text{RC}(B^{k,1} \otimes B^{r',1})$, where we have the arrow $r \xrightarrow{b}$ in D as follows. Let $e_{a_1} e_{a_2} \cdots e_{a_m} b = \bar{\Lambda}_\sigma$, and define $\text{lb}(\nu, J)$ as the rigged configuration obtained by adding a singular row of length 1 to $\nu^{(a_i)}$, for all $1 \leq i \leq m$. By weight considerations, the map is well-defined since the result is independent of the order of the path from b to the highest weight. Note that we can consider $\delta \circ \text{lb}$ to be the same procedure as δ except starting at b .

Next, we define lb on $B^{r,1}$ by requiring that the diagram

$$\begin{array}{ccc} \text{RC}(B^{r,1}) & \xrightarrow{\text{lb}} & \text{RC}(B^{\sigma,1} \otimes B^{r',1}) \\ \Phi \downarrow & & \downarrow \Phi \\ B^{r,1} & \xrightarrow{\text{lb}} & B^{\sigma,1} \otimes B^{r',1} \end{array}$$

commutes, where σ is the unique sink in the lb-diagram. In particular, we note that lb is a strict $U_q(\mathfrak{g}_0)$ -crystal embedding. Therefore, we define *Kirillov–Reshetikhin (KR) tableaux* as the tableaux given by iterating the lb map, where the entries elements in $B^{\sigma,1}$ and the classical crystal structure is induced by the reverse column reading word. See Appendix A for the description of $B^{r,1}$ in types $E_{6,7,8}^{(1)}$ and $E_6^{(2)}$.

For example, consider for $\text{lb}: B^{r,1} \rightarrow B^{\sigma,1} \otimes B^{\sigma',1}$ corresponding to the arrow $r \xrightarrow{b} \sigma$, we can use this to construct the tableau $\begin{bmatrix} x \\ y \end{bmatrix}$, where $x, y \in B^{\sigma,1}$, given by its image under lb , which is $\begin{bmatrix} y \\ x \end{bmatrix}$. We also note that the construction of the KR tableaux is dependent upon the choice of lb-diagram.

We then extend the left-box map to $\text{lb}: B^{r,1} \otimes B^\bullet \rightarrow B^{k,1} \otimes B^{r',1} \otimes B^\bullet$, with respect to the lb-diagram D , as the strict $U_q(\mathfrak{g}_0)$ -crystal embedding given by $b \otimes b_\bullet \mapsto \text{lb}(b) \otimes b_\bullet$.

We note that this is a generalization of the lb map for the KSS bijection. Specifically, for the non-exceptional types, the defining lb-diagram is

$$1 \xleftarrow{\boxed{1}} 2 \xleftarrow{\boxed{2}} \cdots \xleftarrow{\boxed{n-2}} n-1 \xleftarrow{\boxed{n-1}} n. \quad (5.1)$$

For type $E_6^{(1)}$, we use the lb-diagram

$$\begin{array}{ccccccc} & & & & \overline{16} & & \\ & & & & \swarrow & & \\ & & 6 & \xleftarrow{\overline{26}} & 2 & \xleftarrow{\overline{25}} & 5 \\ & & \swarrow & & & & \\ 1 & \xleftarrow{\overline{13}} & 3 & \xleftarrow{\overline{34}} & 4 & & \end{array} \quad (5.2)$$

Remark 5.1. We could alternatively use the lb-digram for type $E_6^{(1)}$ by having the edge $3 \xleftarrow{\overline{23}} 2$ instead of $6 \xleftarrow{\overline{26}} 2$. However, this results in different KR tableaux.

Using δ_6 , we define the lb^\vee -diagram for type $E_6^{(1)}$ by

$$\begin{array}{ccccccc} & & & & \overline{12} & & \\ & & & & \swarrow & & \\ & & 1 & \xleftarrow{\overline{12}} & 2 & \xleftarrow{\overline{23}} & 3 \\ & & \swarrow & & & & \\ 6 & \xleftarrow{\overline{56}} & 5 & \xleftarrow{\overline{45}} & 4 & & \end{array} \quad (5.3)$$

For type $E_7^{(1)}$, the definition of left-box is given by the lb-diagram

$$\begin{array}{ccccccc} & & & & \overline{12} & & \\ & & & & \swarrow & & \\ & & 1 & \xleftarrow{\overline{12}} & 2 & \xleftarrow{\overline{23}} & 3 \\ & & \swarrow & & & & \\ 7 & \xleftarrow{\overline{67}} & 6 & \xleftarrow{\overline{56}} & 5 & \xleftarrow{\overline{45}} & 4 \end{array} \quad (5.4)$$

Example 5.2. In type $E_7^{(1)}$ for $B^{4,1}$, consider the rigged configuration

$$(\nu, J) = 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 1 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 1 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 .$$

Note that $e_7 e_6 e_5(4\bar{5}) = 7$, and so we have

$$\text{lb}(\nu, J) = 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 1 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 1 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0$$

in $\text{RC}(B^{7,1} \otimes B^{5,1})$. Next, by applying δ_7 , we remove the following boxes:

$$0 \begin{array}{|c|} \hline \ell_7 \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \ell_5 \\ \hline \end{array} 0 \quad 1 \begin{array}{|c|} \hline \ell_6 \\ \hline \square \\ \hline \end{array} 1 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \ell_8 \\ \hline \ell_4 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \ell_9 \\ \hline \ell_3 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \ell_{10} \\ \hline \ell_2 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \ell_{11} \\ \hline \ell_1 \\ \hline \end{array} 0 .$$

Thus δ_7 returns $3\bar{7}$ and the resulting rigged configuration $(\delta_7 \circ \text{lb})(\nu, J) \in \text{RC}(B^{5,1})$ is the following:

$$\emptyset \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad \emptyset .$$

Since $e_7 e_6(5\bar{6}) = 7$, applying lb results in

$$\emptyset \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline 0 \\ \hline \end{array} 0 ,$$

and applying δ_7 selects

$$\emptyset \quad 0 \begin{array}{|c|} \hline \ell_5 \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \ell_6 \\ \hline 0 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \ell_7 \\ \hline \ell_4 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \ell_8 \\ \hline \ell_3 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \ell_9 \\ \hline \ell_2 \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \ell_{11} \\ \hline \ell_1 \\ \hline \end{array} 0 ,$$

which yields the empty rigged configuration and a return value of $1\bar{6}7$. Thus, iterating this, we have

$$\Phi(\nu, J) = \begin{array}{|c|} \hline 7 \\ \hline \bar{7}6 \\ \hline 1\bar{6}7 \\ \hline \bar{1}3\bar{7} \\ \hline \end{array} .$$

For type $E_8^{(1)}$, the lb -diagram is

$$\begin{array}{ccccccc} & & & \bar{1}2 & & \bar{2}3 & \\ & & & \longleftarrow & & \longleftarrow & \\ & & & 1 & 2 & 3 & \\ & \nearrow & & & & & \\ \bar{1}8 & & & & & & \\ \longleftarrow & & & & & & \\ 8 & 7 & 6 & 5 & 4 & & \\ \bar{7}8 & \bar{6}7 & \bar{5}6 & \bar{4}5 & & & \end{array} . \quad (5.5)$$

For type $E_6^{(2)}$, we use the lb -diagram

$$\begin{array}{ccc} & & 4 \\ & & \nearrow \\ & & \bar{1}4 \\ \longleftarrow & & \longleftarrow \\ 1 & 2 & 3 \\ \bar{1}2 & \bar{2}3 & \\ & 22 & \end{array} . \quad (5.6)$$

5.1. Onwards and upwards. We have shown that δ induces a bijection Φ . Now we need to show lb induces a bijection. Let σ denote the unique sink of the lb -diagram.

Lemma 5.3. *The vacancy numbers are invariant under lb .*

Proof. Given an arrow $r \xrightarrow{b} r'$, note that $\text{wt}(b) = \overline{\Lambda}_r - \overline{\Lambda}_{r'}$. Since lb adds rows of length 1, and the number of rows in $\nu^{(a)}$ corresponds to the coefficient of α_a in $\overline{\Lambda}_\sigma - \text{wt}(b) = \overline{\Lambda}_\sigma - \overline{\Lambda}_r + \overline{\Lambda}_{r'}$, the claim follows from the definition of vacancy numbers. \square

Proposition 5.4. *Suppose $r \neq \sigma$, and let $r \rightarrow r'$ be the outgoing arrow of r in the lb -diagram. Then we have*

$$\text{lb} \circ \Phi = \Phi \circ \text{lb}.$$

Proof. Consider $r \xrightarrow{b_r} r'$ in the lb -diagram. Suppose the claim holds for r' . Suppose we have $b \otimes b_\bullet \in B^{r,1} \otimes B^\bullet$ such that b is classically highest weight. Let $b_r \otimes b' \otimes b_\bullet = \text{lb}(b \otimes b_\bullet)$, and we note that b' is classically highest weight and $\varepsilon_a(b_r) \leq \varphi_a(b')$ for all $a \in I_0$. Therefore, after adding b' , there are no singular (resp. quasisingular) rows in $\nu^{(a)}$ for all a such that $\varepsilon_a(b_r) = 1$ (resp. $\varepsilon_a(b_r) = 2$). Hence, when we add b_r , we only add singular rows of length 1, and we can apply lb^{-1} to the result.

Next, we proceed by induction on the path to the classically highest weight element. Suppose $f_a(b_r \otimes b') = f_a b_r \otimes b'$, then the claim follows by induction as one additional box is added by δ_σ , but it could not have changed the length 1 singular rows added. Suppose $f_a(b_r \otimes b') = b_r \otimes f_a b'$, and suppose this adds a singular string to $\nu^{(a)}$ such that lb^{-1} is undefined. The proof that the claim holds is similar to the proof of [Sch05, Thm. 4.1]. \square

Lemma 5.5. *Let $B = B^{r,1} \otimes \bigotimes_{i=1}^N B^{r_i, s_i}$. For $(\nu, J) \in \text{RC}(B)$, we have*

$$\text{cc}(\text{lb}(\nu, J)) - \text{cc}(\nu, J) = \sum_{k=1}^{\ell} t_{a_k}^\vee \sum_{j \in \mathbb{Z}_{>0}} L_j^{(a_k)} + \frac{1}{2} \sum_{k=1}^{\ell} t_{a_k}^\vee (\delta_{a_k r'} + \delta_{a_k \sigma} - \delta_{a_k r}),$$

where we have the path $b = f_{a_1} f_{a_2} \cdots f_{a_\ell} u_{\Lambda_\sigma}$ for the arrow $r \xrightarrow{b} r'$ in the lb -diagram.

Proof. Recall from Equation (4.5) that the cocharge of a configuration can be expressed as

$$\text{cc}(\nu) = \frac{1}{2} \sum_{(a,i) \in \mathcal{H}_0} t_a^\vee \left(\sum_{j \in \mathbb{Z}_{>0}} L_j^{(a)} \min(i, j) - p_i^{(a)} \right) m_i^{(a)}.$$

Since lb adds a singular string of length 1 to $\nu^{(a_k)}$ for all $1 \leq k \leq \ell$, where we have $b = f_{a_1} f_{a_2} \cdots f_{a_\ell} u_\theta$, for the arrow $r \xrightarrow{b} r'$ in the corresponding lb -diagram. A straightforward tedious calculation gives

$$\begin{aligned} K &:= \sum_{(a,i) \in \mathcal{H}_0} t_a^\vee \sum_{j \in \mathbb{Z}_{>0}} \left(\tilde{L}_j^{(a)} \tilde{m}_i^{(a)} - L_j^{(a)} m_i^{(a)} \right) \min(i, j) \\ &= \sum_{(a,i) \in \mathcal{H}_0} t_a^\vee (\delta_{a r'} + \delta_{a \sigma} - \delta_{a r}) m_i^{(a)} + \sum_{k=1}^{\ell} t_{a_k}^\vee \sum_{j \in \mathbb{Z}_{>0}} L_j^{(a_k)} \\ &\quad + \sum_{k=1}^{\ell} t_{a_k}^\vee (\delta_{a_k r'} + \delta_{a_k \sigma} - \delta_{a_k r}) \\ &= - \sum_{k=1}^{\ell} t_{a_k}^\vee p_1^{(a_k)} + 2 \sum_{k=1}^{\ell} t_{a_k}^\vee \sum_{j \in \mathbb{Z}_{>0}} L_j^{(a_k)} + \sum_{k=1}^{\ell} t_{a_k}^\vee (\delta_{a_k r'} + \delta_{a_k \sigma} - \delta_{a_k r}). \end{aligned}$$

Note that during the derivation, we used Lemma 5.3 and note that $\min(i, j) = 1$ if either $i = 1$ or $j = 1$. Therefore, we have

$$\begin{aligned} \text{cc}(\text{lb}(\nu, J)) - \text{cc}(\nu, J) &= \frac{1}{2} \left(K - \sum_{k=1}^{\ell} t_{a_k}^{\vee} p_1^{(a_k)} \right) + \sum_{k=1}^{\ell} t_{a_k}^{\vee} p_1^{(a_k)} \\ &= \sum_{k=1}^{\ell} t_{a_k}^{\vee} \sum_{j \in \mathbb{Z}_{>0}} L_j^{(a_k)} + \frac{1}{2} \sum_{k=1}^{\ell} t_{a_k}^{\vee} (\delta_{a_k r'} + \delta_{a_k \sigma} - \delta_{a_k r}) \end{aligned}$$

as desired. \square

We first need a type-independent proof of [ST12, Thm. 6.1], which we show for $s = 1$.

Proposition 5.6. *Let $B = \bigotimes_{i=1}^N B^{r_i, 1}$ for dual untwisted types. Let $\lambda = -\sum_{i=1}^N \Lambda_{\xi(r_i)}$. Let t_{λ} denote the translation by λ in the extended affine Weyl group, and let $t_{\lambda} = v\tau$, where v is an element in the affine Weyl group. Then there exists a $U'_q(\mathfrak{g})$ -crystal isomorphism*

$$j: B(\Lambda_{\tau(0)}) \rightarrow B \otimes B(\Lambda_0)$$

such that the image of the Demazure subcrystal $B_v(\Lambda_{\tau(0)})$ is $B \otimes u_{\Lambda_0}$.

Proof. Recall from [Lit95a, Lit95b] that any Littelmann path that stays in the dominant chamber with an endpoint of λ generates $B(\lambda)$. The claim follows from using (projected) level-zero LS paths [NS06, NS08a, NS08b], that tensor products in the Littelmann path model corresponding to concatenation of the LS paths [Lit95a, Lit95b]. \square

Lemma 5.7. *Let $B = \bigotimes_{i=1}^N B^{r_i, 1} \otimes B^{r, 1}$. For $b \in B$, we have*

$$D(\text{rb}(b)) - D(b) = \sum_{k=1}^{\ell} t_{a_k}^{\vee} \left(\sum_{j \in \mathbb{Z}_{>0}} L_j^{(a_k)} + \frac{1}{2} (\delta_{a_k r'} + \delta_{a_k \sigma} - \delta_{a_k r}) \right),$$

where $b = f_{a_1} f_{a_2} \cdots f_{a_{\ell}} u_{\overline{\Lambda_{\sigma}}}$, for the arrow $r_N \xrightarrow{b} r'_N$.

Proof. Let u denote the maximal vector in B . Note that $u^* \in B$ is the unique element of classical weight $w_0 \text{wt}(u)$ and is in the same classical component as u .

Let $B^{\bullet} = \bigotimes_{i=1}^N B^{r_i, 1}$, and denote the maximal vector in B by $u_{\bullet} \otimes u_r$, where u_{\bullet} and u_r are the maximal elements in B^{\bullet} and $B^{r, 1}$ respectively. We note that the unique element of classical weight $w_0 \text{wt}(u)$ is given by $v = u_{\bullet}^* \otimes u_r^*$ (technically we apply \star to each factor individually, but we do not care about the ordering of B^{\bullet} and u_{\bullet}). We note that v is in the same classical component as $u_{\bullet} \otimes u_r$. Therefore, v^* is the maximal element in $B^{r, 1} \otimes B^{\bullet}$.

Therefore, we have

$$\begin{aligned} \text{lb}(v^*) &= b \otimes u_{r'} \otimes u_{\bullet}, \\ \text{lb}(v^*)^* &= u_{\bullet}^* \otimes u_{\xi(r')} \otimes b^* =: b_0, \end{aligned}$$

where

$$b^* = f_{\xi(a_1)} f_{\xi(a_2)} \cdots f_{\xi(a_{\ell})} u_{\overline{\Lambda_{\sigma}}}.$$

Next, we want to apply a sequence $e_{\bar{j}_k}$ to b_{k-1} such that we obtain

$$b_k := u_{\bullet}^* \otimes u_{\xi(r')} \otimes (f_{\xi(a_{k+1})} \cdots f_{\xi(a_{\ell})} u_{\overline{\Lambda_{\sigma}}}),$$

where $b_{\ell} = u_{\bullet}^* \otimes u_{\xi(r')} \otimes u_{\overline{\Lambda_{\sigma}}}$.

Fix some $j \in I_0$ and i . Let $I_{r_i} = I \setminus \{r_i\}$. If we have $v_{r_i} \otimes x$, where v_{r_i} is the lowest weight element in $B^{r_i, 1}$, then if $r_i \neq j$, we have $e_j(v_{r_i} \otimes x) = v_{r_i} \otimes e_j x$. If $r_i = j$, then $e_j^2(v_{r_i} \otimes x) = (e_j v_{r_i}) \otimes (e_j x)$, but we note that there exists a sequence of crystal operators $e_{\bar{j}}$ that acts only on the left that returns to $v_i \otimes (e_j x)$ that uses precisely t_{a_k} 0-arrows. This follows from the fact that $e_j v_i$ is the unique I_{r_i} -lowest weight element in the unique I_{r_i} -component $B(\bar{\Lambda}_0) \subseteq B^{r, 1}$. We can these 0-arrows are Demazure as the extra $B^{\sigma, 1}$ results in $\varphi_0(\text{lb}(v^*)^*) - \varphi_0(v) = 1$.

\mathfrak{g}	$B_n^{(1)}$	$C_n^{(1)}$	$F_4^{(1)}$	$G_2^{(1)}$
\mathfrak{g}^\vee	$D_{n+1}^{(1)}$	$D_{n+1}^{(2)}$	$E_6^{(2)}$	$D_4^{(3)}$
r	n	1	4	1
δ_r^v	$\widehat{\delta}_n \circ \widehat{\delta}_{n+1}$	$\widehat{\delta}_1$	$\widehat{\delta}_1 \circ \widehat{\delta}_1 \circ \widehat{\text{lb}}$	$\widehat{\delta}_1 \circ \widehat{\delta}_1 \circ \widehat{\text{lb}}$
$(\gamma_a)_{a \in I}$	$(2 \dots, 2, 1)$	$(2, 1, \dots, 1, 2)$	$(2, 2, 2, 1, 1)$	$(3, 1, 3)$

TABLE 2. The scaling factors and δ_r^v for the virtualization map v given by $\mathfrak{g} \rightarrow \mathfrak{g}^\vee$.

From [ST12, Lemma 7.3], the number of Demazure 0-arrows, a 0-arrow that is not at the end of a 0-string, between two elements determines the difference in energy. Therefore, we have

$$D(\text{rb}(v)) - D(v) = \sum_{k=1}^{\ell} t_{a_k}^\vee \sum_{j \in \mathbb{Z}_{>0}} \widetilde{L}_j^{(a_k)} + S,$$

where \widetilde{L} corresponds to B^\bullet and S is the claim for a single tensor factor $B^{r,1}$. Thus, it remains to show the claim for a single tensor factor, which is a finite computation using the results from [LNS⁺15, LNS⁺16b]. Thus the claim holds for any element in the same classical component of v .

Let $d(b)$ denote the affine grading of b [ST12, Def. 7.1], the number of Demazure 0-arrows in the path from b to $u_{\Lambda_{\tau(0)}}$ in $B(\Lambda_{\tau(0)})$. Next, for any other classically lowest weight element v' , we have $d(\text{rb}(v')) = d(b')$ since the number of Demazure 0-arrows is determined by the coefficient of α_0 in $\Lambda_{\tau(0)} - \text{wt}(v')$ and $\text{wt}(\text{rb}(v')) = \text{wt}(v')$. Therefore, we have

$$\begin{aligned} D(\text{rb}(v')) - D(v') &= [d(\text{rb}(v')) - d(v_{\text{rb}})] - [d(v') - d(v)] \\ &= [d(\text{rb}(v')) - d(v')] + [d(v) - d(v_{\text{rb}})] \\ &= D(\text{rb}(v)) - D(v), \end{aligned}$$

where v_{rb} is the minimal element in $\text{rb}(B)$. Hence, the claim follows. \square

We first note that the proof of Lemma 5.5 and Lemma 5.7 is true for general type \mathfrak{g} and any lb-diagram. Furthermore, the proof of Lemma 5.7 should essentially work for general tensor factors and provided there is an analog of [ST12, Lemma 7.2] and [ST12, Thm. 6.1] for all $B^{r,s}$ in any type. In particular, we are using [ST12, Lemma 7.3], which is essentially a type-independent proof (other than the single factor case, i.e., [ST12, Lemma 7.2]).

6. UNTWISTED TYPES

Let \mathfrak{g} be of type $C_n^{(1)}$, $F_4^{(1)}$, or $G_2^{(1)}$. For these types, we note that there is a virtualization map v to the corresponding dual type \mathfrak{g}^\vee and that the scaling factors $(\gamma_a)_{a \in I}$ are exactly those considering \mathfrak{g} as a folding of the corresponding simply-laced type $\widehat{\mathfrak{g}}$. For type $G_2^{(1)}$ to $D_4^{(3)}$, we also need to interchange $1 \leftrightarrow 2$ due to our numbering conventions. However, for type $B_n^{(1)}$, we will use the embedding into type $D_{n+1}^{(1)}$ as it affords an easier proof than $A_{2n-1}^{(2)}$. Using this, we construct the bijection Φ by showing it commutes with the virtualization map to the dual untwisted type.

6.1. The map δ_r . It is known that $B^{r,1}$ can be realized as a virtual crystal inside of

$$\widehat{B}^{r,1} = \begin{cases} B^{n+1,1} \otimes B^{n,1} & \text{if } \mathfrak{g} = B_n^{(1)}, \\ B^{2,1} & \text{if } \mathfrak{g} = G_2^{(1)}, \\ B^{r,1} & \text{otherwise,} \end{cases}$$

of type \mathfrak{g}^\vee . We want to define the map $\delta_r := v^{-1} \circ \delta_r^v \circ v$, where r and δ_r^v are given in Table 2. Thus, we need to show that

$$\delta_r^v : \widehat{\text{RC}}(B^{r,1} \otimes B^\bullet) \rightarrow \widehat{\text{RC}}(B^\bullet)$$

is well-defined when restricted to the image of v .

Theorem 6.1. Suppose $(\widehat{\nu}, \widehat{J})$ satisfies Equation (2.9), then $\delta_r^v(\widehat{\nu}, \widehat{J})$ satisfies Equation (2.9). Moreover, the map δ_r^v is well-defined when restricted to the image of v .

Proof. We proceed by induction by examining $(\delta_r^v)^{-1}$, where the base case is done by $\delta_r^v(\widehat{\nu}, \widehat{J}) = (\widehat{\nu}, \widehat{J})$, which returns the highest weight element $v(u_{\overline{\Lambda}_r})$. Next we assume the claim holds when δ_r^v returns $\widehat{b} := v(b)$. Fix $a \in I_0$. Let $(\widehat{\nu}', \widehat{J}')$ be the rigged configuration such that $(\widehat{\nu}, \widehat{J}) := \delta_r^v(\widehat{\nu}, \widehat{J}) = \delta_r^v(\widehat{\nu}', \widehat{J}')$ but with a return value of $f_a^v \widehat{b} = v(f_a b)$.

For type $C_n^{(1)}$, we have that $(\widehat{\nu}', \widehat{J}')$ differs from $(\widehat{\nu}, \widehat{J})$ by the addition of γ_a boxes to a row in $\widehat{\nu}^{(a)}$. From Equation (2.10), we have all riggings \widehat{J}' are still multiples of $\gamma_{a'}$ for all $a' \in \widehat{I}$, and the claim follows.

Next we consider type $F_4^{(1)}$. The case when $f_a^v \text{lb}(\widehat{b}) = (f_a^v \widehat{b}_2) \otimes \widehat{b}_1$ is similar to the type $C_n^{(1)}$ case. Now suppose $f_a^v \text{lb}(\widehat{b}) = (\widehat{f}_a \widehat{b}_2) \otimes (\widehat{f}_a \widehat{b}_1)$. Note that $\widehat{\delta}^{-1}$ for $\widehat{f}_a \widehat{b}_1$ starts at $\nu^{(a)}$ and the only singular rows in $\nu^{(a')}$ for $\gamma_{a'} > 1$ are the rows selected by $\widehat{\delta}_r^{-1}$ by Equation (2.10). Hence, applying $\widehat{\delta}^{-1}$ for $\widehat{f}_a \widehat{b}_2$ must select those same rows in $\nu^{(a')}$ for $\gamma_{a'} > 1$ as there are sufficient singular rows in $\nu^{(a')}$ for $\gamma_{a'} = 1$ of length $\ell_{i_j} \leq \ell_k \leq \ell_{i_{j+1}}$ for all $i_j \leq k \leq i_{j+1}$, where $\ell_{i_1}, \dots, \ell_{i_q}$ are the lengths of the rows selected of $\nu^{(a')}$ for fixed a' such that $\gamma_{a'} > 1$. We note that such rows exists because $\widehat{b}_2 \geq \widehat{b}_1$. Once all such rows have been paired, we are equivalent to the case of $\widehat{b}' \otimes v(u_{\overline{\Lambda}_r})$ with all sufficiently long rows non-singular. Hence, the claim follows by induction.

Now suppose $f_a^v \text{lb}(\widehat{b}) = \widehat{b}_2 \otimes (f_a^v \widehat{b}_1)$ and let $(\widetilde{\nu}, \widetilde{J})$ and $(\widetilde{\nu}', \widetilde{J}')$ denote $\widehat{\delta}_r^{-1}(\widehat{\nu}, \widehat{J})$ by adding \widehat{b}_1 and $f_a^v \widehat{b}_1$ respectively. Note that any row selected to obtain $\widetilde{\nu}'$ is at most as long as that to obtain $\widetilde{\nu}$ and that $\widehat{\delta}_r^{-1}$ added γ_a boxes to this row in obtaining $(\widetilde{\nu}')^{(a)}$. Therefore, this case follows from our induction assumption for the case where the necessary rows are made to be (non-)singular but with a return value of $\text{lb}(\widehat{b})$.

The proof for type $G_2^{(1)}$ is similar. For type $B_n^{(1)}$, we note that if $\widehat{f}_a(\widehat{b}_2 \otimes \widehat{b}_1) = (\widehat{f}_a \widehat{b}_2) \otimes \widehat{b}_1$ for $a \neq n, n+1$, then we must have previously had \widehat{f}_a act on the right. Specifically, this is equivalent to having s_a have the same sign in both columns of $B(\widehat{\Lambda}_n) \otimes B(\widehat{\Lambda}_{n+1})$. Thus the proof is also similar for type $B_n^{(1)}$. \square

We note that our proof is almost type independent as it is the same general technique, but we require some mild type dependencies. We also note that Theorem 6.1, [SS15b, Rem. 5.15], and [SS15b, Thm. 6.5] implies that we could define δ by considering the virtualization map of $B_n^{(1)}$ into $A_{2n-1}^{(1)}$.

Remark 6.2. Instead of using the scaling factors to enlarge the partitions, we could instead consider scaling each $\nu^{(a)}$ by $1/T_a$. So that the partitions have integer lengths, we scale by (a multiple of) $\max_a T_a$, which the net effect would be to multiply by γ_a . This suggests a strong relationship between the Weyl chamber geometry and rigged configurations through the bijection Φ .

6.2. Defining lb and general columns. The lb-diagram for type $C_n^{(1)}$ is given by Equation (7.2). For type $B_n^{(1)}$, the lb-diagram we consider is

$$\begin{array}{c} \nearrow n-1 \\ (n-1)\overline{n} \nearrow \\ \vdots \\ n \xleftarrow{1\overline{n}} 1 \end{array}.$$

For type $F_4^{(1)}$, we use the lb-diagram

$$\begin{array}{c} \nearrow 1 \\ 1\overline{4} \nearrow \\ \vdots \\ 4 \xleftarrow{3\overline{4}} 3 \xleftarrow{2\overline{3}} 2 \end{array}.$$

For type $G_2^{(1)}$, we want to consider $B(\overline{\Lambda}_1)$ as a virtual crystal inside of $B(3\overline{\Lambda}_1) \subseteq B(\overline{\Lambda}_2) \otimes B(\overline{\Lambda}_2)$. This corresponds to adding a singular row of length 1 to $\nu^{(2)}$, which would be of length γ_2 in $\gamma_2 \nu^{(2)}$. This allows us to construct an lb-diagram as $1 \xrightarrow{1^3 \overline{2}} 2$.

7. MAIN RESULTS

We gather our main results here.

7.1. Statistic preserving bijection.

Proposition 7.1. *Let \mathfrak{g} be of type $A_n^{(1)}$, $D_n^{(1)}$, $E_{6,7,8}^{(1)}$, or $E_6^{(2)}$. Let $B = \bigotimes_{i=1}^N B^{r_i,1}$ and $B' = \bigotimes_{i=1}^{N'} B^{r'_i,1}$ (except possibly for r_i and r'_i being 4, 5 in type $E_8^{(1)}$). Then the diagram*

$$\begin{array}{ccc} \mathrm{RC}(B \otimes B') & \xrightarrow{\Phi} & B \otimes B' \\ \mathrm{id} \downarrow & & \downarrow R \\ \mathrm{RC}(B' \otimes B) & \xrightarrow{\Phi} & B' \otimes B \end{array}$$

commutes.

Proof. For type $A_n^{(1)}$ (resp. $D_n^{(1)}$) was shown in [KSS02] (resp. [OSSS16]). For types $E_{6,7,8}^{(1)}$ and $E_6^{(2)}$, the claim reduces to $B^{r_1,1} \otimes B^{r_2,1}$ by the definition of Φ and is a finite computation. \square

Now we can show that the map Φ is a bijection that sends cocharge to energy.

Theorem 7.2. *Let $B = \bigotimes_{i=1}^N B^{r_i,1}$ be a tensor product of KR crystals. Then the map*

$$\Phi: \mathrm{RC}(B) \rightarrow B$$

is a bijection on highest weight elements such that $\Phi \circ \eta$ sends cocharge to energy.

Proof. By Proposition 5.4, Lemma 5.5, and Lemma 5.7, showing the bijection is well-defined and preserves statistics is reduced to showing when the left-most factor for $\mathrm{RC}(B)$ and right-most factor for B^{rev} is minuscule or adjoint. Thus the claim follows by Lemma 3.8 and Lemma 4.5. \square

Theorem 7.3. *Let Φ be defined using δ_r such that r is a minuscule node. Then $\Phi: \mathrm{RC}(B) \rightarrow B$ is a $U_q(\mathfrak{g}_0)$ -crystal isomorphism.*

Proof. A straightforward check shows that Φ preserves weights. A tedious but straightforward check shows that the arguments in [Sak14] extend to all minuscule nodes by the description of δ_r and that the arguments are about the relative position where two boxes are added by applying f_a . \square

For $r = 1, 2, 6$ in type $E_6^{(1)}$, we can describe e_0 and f_0 on $\mathrm{RC}(B^{r,s})$ by using the description in [JS10] as it is given solely in terms of ε_i , φ_i , and the weight. Hence, Theorem 7.3 immediately implies the following.

Corollary 7.4. *Let \mathfrak{g} be of type $E_6^{(1)}$ and $r = 1, 2, 6$. Then*

$$\Phi: \mathrm{RC}(B^{r,s}) \rightarrow B^{r,s}$$

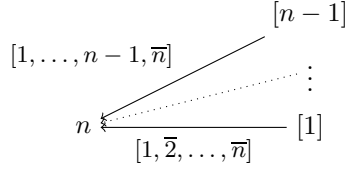
is a $U'_q(\mathfrak{g})$ -crystal isomorphism.

7.2. Virtual bijection. For type $E_6^{(2)}$ as a folding of type $E_6^{(1)}$, we also need the notion of a *virtual lb map*, which we denote by lb^v . Specifically, we generalize the notation of an lb-diagram to use for the map lb^v by also having an arrow $(r_1, r_2) \rightarrow r'$ defines a map $\mathrm{lb}^v: B^{r_1,1} \otimes B^{r_2,1} \rightarrow B^{1,1} \otimes B^{r',1}$. Thus, we define lb^v by using the virtual lb^v-diagram

$$\begin{array}{ccccc} & & (1, 6) & & \\ & \nearrow \overline{2}16 & & & \\ 2 & \xleftarrow{\quad \overline{2}4 \quad} & 4 & \xleftarrow{\quad \overline{4}35 \quad} & (3, 5) . \end{array} \quad (7.1)$$

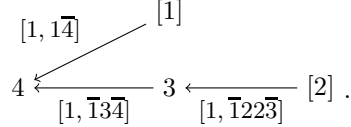
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For type $B_n^{(1)}$ as the dual of $A_{2n-1}^{(2)}$, the lb^v -diagram we consider is



where $[r]$ corresponds to $B^{r,2}$.

For type $F_4^{(1)}$ as the dual of $E_6^{(2)}$, the lb^v -diagram is a modification of Equation (5.6):



For type $G_2^{(1)}$ as the dual of $D_4^{(3)}$, recall that we consider $B(\overline{\Lambda}_1)$ as the natural virtual crystal of $B(3\overline{\Lambda}_1) \subseteq B(\overline{\Lambda}_2) \otimes B(\overline{\Lambda}_2)$ in type $G_2^{(1)}$. Continuing this in type $D_4^{(3)}$, we construct an lb^v -diagram as $2 \xrightarrow{[1,3]} [[1]]$, where $[[1]]$ corresponds to $B^{1,3}$.

Proposition 7.5. *Let v be one of the virtualization maps of*

$$C_n^{(1)} \hookrightarrow A_{2n}^{(2)} \hookrightarrow D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)}, \quad (7.2)$$

with scaling factors $(2, 1, \dots, 1)$, $(1, \dots, 1, 2)$, and $(1, \dots, 1)$ respectively. Then

$$v \circ \delta_\theta = \widehat{\delta}_\theta \circ v.$$

Proof. Let \mathfrak{g} be of type $D_{n+1}^{(2)}$ and $\widehat{\mathfrak{g}}$ of type $A_{2n-1}^{(2)}$. We note that if δ_θ selects a singular string in $\nu^{(a)}$ for $a \neq n$, then it must select the same singular string in $\nu^{(a')} = \nu^{(a'')}$ for all $a', a'' \in \phi^{-1}(a)$.

For the remaining virtualization maps, this follows from the description of δ_θ as per Section 4.3. \square

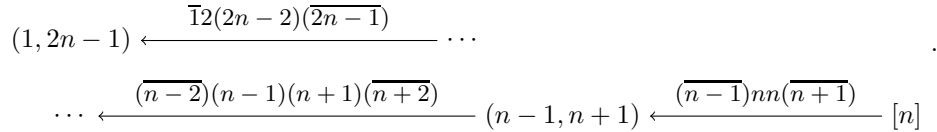
Remark 7.6. We can also compose the virtualization maps of Equation (7.2), and we obtain another proof of [OSS03c, Thm. 7.1].

We also can use

$$C_n^{(1)} \hookrightarrow A_{2n}^{(2)\dagger} \hookrightarrow D_{n+1}^{(2)} \hookrightarrow A_{2n-1}^{(1)},$$

with scaling factors $(1, \dots, 1, 2)$, $(2, 1, \dots, 1)$, and $(1, \dots, 1)$ respectively, instead of Equation (7.2).

For type $D_{n+1}^{(2)}$ as a folding of type $A_{2n-1}^{(1)}$, we use the lb^v -diagram:



Lemma 7.7. *Let \mathfrak{g} be of non-simply-laced affine type. Then we have*

$$v \circ \text{lb} = \text{lb}^v \circ v,$$

where v is one of the virtualization maps given above.

Proof. Suppose \mathfrak{g} is of type $E_6^{(2)}$. Recall that lb adds a length 1 singular string to $\widehat{\nu}^{(a_i)}$, where $e_{a_1} \cdots e_{a_k} b = u_{\overline{\Lambda}_2}$. Moreover, recall that lb does not change the vacancy numbers. It is a finite computation to show that for any arrow $r \xrightarrow{b} r'$ in the lb -diagram in Equation (5.6), we have an arrow $\phi^{-1}(r) \xrightarrow{v(b)} \phi^{-1}(r')$, where $\phi: E_6^{(1)} \searrow E_6^{(2)}$ is the diagram folding, in the lb^v -diagram in Equation (7.1). Thus the claim follows.

For the other types, the proof is similar with also considering the doubling (or tripling) map. \square

Theorem 7.8. *Let \mathfrak{g} be of non-simply-laced affine type. Then*

$$v \circ \Phi = \widehat{\Phi} \circ v,$$

where v is one of the virtualization maps given above. Moreover, Φ is a bijection such that $\Phi \circ \eta$ sends cocharge to energy.

Proof. Lemma 7.7 implies that it is sufficient to show $v \circ \delta = \widehat{\delta} \circ v$. For untwisted types, this follows by definition and Theorem 6.1. For all dual untwisted types, type $A_{2n}^{(2)}$, and type $A_{2n}^{(2)\dagger}$, the proof is similar to the proof of Proposition 7.5. Thus the claim follows. \square

Proposition 7.9. *Let \mathfrak{g} be of affine type. Let $B = \bigotimes_{i=1}^N B^{r_i,1}$ and $B' = \bigotimes_{i=1}^{N'} B^{r'_i,1}$. Then the diagram*

$$\begin{array}{ccc} v(B \otimes B') & \xrightarrow{\widehat{R}} & v(B \otimes B') \\ \uparrow v & & \uparrow v \\ B' \otimes B & \xrightarrow{R} & B' \otimes B \end{array}$$

commutes. Moreover, the combinatorial R -matrix can be defined as the restriction of \widehat{R} to the image of v .

Proof. This follows from Proposition 5.6. \square

Corollary 7.10. *Proposition 7.1 holds for all affine types.*

Proof. This follows from Theorem 7.8, Proposition 7.9, and Proposition 7.1. \square

7.3. Higher levels for minuscule nodes.

Proposition 7.11. *Let r be such that $\overline{\Lambda}_r$ is a minuscule weight. Then $B(s\overline{\Lambda}_r) \subseteq B(\overline{\Lambda}_r)^{\otimes s}$ is characterized by*

$$\{b_1 \otimes \cdots \otimes b_s \mid b_1 \leq \cdots \leq b_s\}.$$

Proof. By [JS10, Lemma 2.9], it is sufficient to show when $s = 2$. We show this by induction, where the base case is $x_1 = x_2 = u_{\overline{\Lambda}_r}$. Next, suppose $x_1 \leq x_2$, and let $x_2 = f_{a_m} \cdots f_{a_1} x_1$. Fix some $i \in I_0$. Consider when $f_i(x_1 \otimes x_2) = x_1 \otimes (f_i x_2)$, then we have $x_1 \leq f_i x_2$. Now, consider the case $f_i(x_1 \otimes x_2) = (f_i x_1) \otimes x_2$. Since $B(\overline{\Lambda}_r)$ is minuscule, all elements are parameterized by trailing words of the longest coset representative $w \in \overline{W}/\overline{W}_{\widehat{r}}$ and that w is fully commutative [Ste01, Prop. 2.1]. Since w is fully commutative, we have must have $f_{a_1} f_i x = f_i f_{a_1} x$, and similarly, there exists a k such that $a_k = i$ and

$$x_2 = f_{a_m} \cdots f_{a_{k+1}} f_{a_{k-1}} \cdots f_{a_1} f_{a_k} x_1.$$

Therefore, we have $f_i x_1 \leq x_2$. \square

Next, we note that the proof given in [SS06a, Lemma 8.2] that the left-split map $\text{ls}: B^{r,s} \otimes B^\bullet \rightarrow B^{r,1} \otimes B^{r,s-1} \otimes B^\bullet$ is preserved under Φ is straightforward to generalize to when r is a minuscule node. We also can show that Φ send the combinatorial R -matrix to the identity map on rigged configurations

Proposition 7.12. *Let $B = \bigotimes_{i=1}^N B^{r_i,s}$ and $B' = \bigotimes_{i=1}^{N'} B^{r'_i,s}$, where r_i and r'_i are minuscule nodes for all i . Then the diagram*

$$\begin{array}{ccc} \text{RC}(B \otimes B') & \xrightarrow{\Phi} & B \otimes B' \\ \text{id} \downarrow & & \downarrow R \\ \text{RC}(B' \otimes B) & \xrightarrow{\Phi} & B' \otimes B \end{array}$$

commutes.

Proof. By the description of Φ it is sufficient to reduce this to the case when $B = B^{r,s}$ and $B' = B^{r',s}$. Recall that $B^{r,s} \cong B(s\overline{\Lambda}_r)$ as $U_q(\mathfrak{g}_0)$ -crystal when r is a special node. Since $\overline{\Lambda}_r$ and $\overline{\Lambda}_{r'}$ are minuscule weights, it follows from [Ste03] that $B^{r,s} \otimes B^{r',s} \cong B(s\overline{\Lambda}_r) \otimes B(s\overline{\Lambda}_{r'})$ is multiplicity free. \square

Next, it is straightforward to see that on rigged configurations, we have rb acting as the identity map on the configuration ν and preserves the corrigings. We also have the following.

Proposition 7.13. *We have*

$$\delta^* \circ \Phi = \Phi \circ \delta^*.$$

Proof. Recall that δ commutes with the classical crystal operators by Theorem 7.3. Note that we can define δ^* by removing the rightmost factor and then going to the highest weight element by [SS06a, Prop. 5.9]. Hence, the claim follows. \square

We also have the following from Proposition 7.13 and that $[\text{lb}, \text{rb}] = 0$.

Corollary 7.14. *We have $[\delta, \delta^*] = 0$.*

The remaining proof of [SS06a, Thm. 8.6] that $\diamond \circ \Phi = \Phi \circ \theta$ holds. Hence, the proof of [SS06a, Thm. 8.8] by using Theorem 7.3. Therefore, we have the following.

Theorem 7.15. *Let $B = \bigotimes_{i=1}^N B^{r_i, s_i}$ be a tensor product of KR crystals, where r_i is a minuscule node for all i . Then the map*

$$\Phi: \text{RC}(B) \rightarrow B$$

is a $U_q(\mathfrak{g}_0)$ -crystal isomorphism such that $\Phi \circ \eta$ sends cocharge to energy.

8. EQUIVALENT BIJECTIONS

In this section, we show that all of our defined bijections give the same bijection and $\Phi = \tilde{\Phi}$ for all types except $B_n^{(1)}$ (where it remains a conjecture), $F_4^{(1)}$, and $G_2^{(1)}$. We show that Φ defined using a combination of various δ_σ defines the same bijection as $\tilde{\Phi}$ for types $A_n^{(1)}$, $D_n^{(1)}$, and $E_6^{(1)}$. Note that for type $A_n^{(1)}$, the map δ_{n-1} is the dual bijection of [SS06a, Sec. 9]. For types $A_n^{(1)}$ and $D_n^{(1)}$, we will show that the bijection defined by δ_σ and $\tilde{\delta}$ are equal. For type $A_{2n-1}^{(2)}$, we only have $\delta_1 = \tilde{\delta}$, which was done in [OSS03a]. For type $D_{n+1}^{(2)}$, we will use the virtualization map and type $A_{2n-1}^{(1)}$. For type $E_6^{(1)}$, the map $\delta_1 = \tilde{\delta}$ from [OS12] and δ_6 follows from the diagram automorphism given by $1 \leftrightarrow 6$.

We will first show that all of the bijections defined using δ_r for any $r \in I_0$ in type $A_n^{(1)}$ are equal.

Lemma 8.1. *Let \mathfrak{g} be of type $A_n^{(1)}$ and $B = B^{r,1} \otimes B^\bullet$. We have*

$$\delta_r = \tilde{\delta} \circ (\tilde{\delta} \circ \text{lb})^{n-1}.$$

Proof. We note that $\delta_1 = \tilde{\delta}$. Thus fix an $r > 1$. We proceed by induction, and note that it is sufficient to show

$$\delta_r = \delta_{r-1} \circ \tilde{\delta} \circ \text{lb} \tag{8.1}$$

by our induction assumption.

Let b_r denote the return value of δ_r and $b_1 \otimes b_{r-1}$ be the return values under $\delta_{r-1} \circ \tilde{\delta} \circ \text{lb}$. Suppose $b_r = u_{\overline{\Lambda}_r}$, and so $\delta_r(\nu, J) = (\nu, J)$ as there are no singular strings in $\nu^{(r)}$. Moreover, it is straightforward to see that $b_1 = \boxed{r}$ and $b_{r-1} = u_{\overline{\Lambda}_{r-1}}$. Hence Equation (8.1) holds in this case. By induction, it is sufficient to show that Equation (8.1) holds for $f_a b_r = f_a(b_1 \otimes b_{r-1})$. Let (ν, J) and (ν', J') be the rigged configurations such that $\delta_r(\nu, J) = \delta_r(\nu', J')$ with return values b_r and $f_a b_r$ respectively.

Suppose $f_a(b_1 \otimes b_{r-1}) = b_1 \otimes (f_a b_{r-1})$. Hence $\varepsilon_a(b_1) = 0$ by Lemma 3.2 and the tensor product rule. Thus, the difference between $\tilde{\delta}(\nu, J)$ and $\tilde{\delta}(\nu', J')$ is given by Lemma 3.5. It follows that $\tilde{\delta}$ must not have selected a row in $\nu^{(a)}$ last and it is singular in $\tilde{\delta}(\nu', J')$. From the definition of δ_{r-1} and δ_r , Equation (8.1) holds.

Suppose $f_a(b_1 \otimes b_{r-1}) = (f_a b_1) \otimes b_{r-1}$. By Lemma 3.2, we must have $\varepsilon_a(b_1) = 0$ since $\varphi_a(b_1) = 1$, and hence $\varphi_a(b_{r-1}) = 0$. Therefore $\tilde{\delta}(\nu, J)$ only differs from $\tilde{\delta}(\nu', J')$ by possibly singular rows in $\nu^{(a)}$ at least as long as $\ell - 1$, where ℓ is the length of the last row selected by $\tilde{\delta}$ in (ν', J') . Therefore, from the definition of δ_{r-1} , Equation (8.1) holds. \square

In order to show that the bijections agree for type $D_n^{(1)}$, we recall the map $\delta_{\text{sp}}: B^{r,1} \otimes B^\bullet \rightarrow B^\bullet$ for $r = n-1, n$ from [Sch05]. Let $v: B^{r,1} \rightarrow B^{r,2}$ denote the virtualization map given by $\gamma_a = 2$ for all $a \in I$. Let $\text{lb}^{(r)}: B^{r,2} \otimes B^\bullet \rightarrow B^{1,1} \otimes B^{n-1,1} \otimes B^{n,1} \otimes B^\bullet$ be given on rigged configurations by adding a length 1 singular row to $\nu^{(a)}$ for all $a \leq n-2$ and $a = n, n-1$ if $r = n-1, n$ respectively. Let $\overline{\text{lb}}: B^{n-1,1} \otimes B^{n,1} \otimes B^\bullet \mapsto B^{n-2,1} \otimes B^\bullet$ be given on rigged configurations by adding a length 1 singular row to $\nu^{(a)}$ for all $a \leq n-2$. We then define $\delta_{\text{sp}}: B^{r,1} \otimes B^\bullet \rightarrow B^\bullet$ by

$$\delta_{\text{sp}} := v^{-1} \circ (\tilde{\delta} \circ \text{lb})^{n-2} \circ \tilde{\delta} \circ \overline{\text{lb}} \circ \tilde{\delta} \circ \text{lb}^{(r)} \circ v.$$

We could also define δ_{sp} by using the (generalized) lb-diagram of

$$1 \xleftarrow{\boxed{1}} \cdots \xleftarrow{\boxed{n-2}} n-2 \xleftarrow{\boxed{n-1}} (n-1, n) \xleftarrow{\boxed{x}} [r],$$

where $x = \bar{n}, n$ if $r = n-1, n$ respectively and recall $[r]$ corresponds to $B^{r,2}$.

Lemma 8.2. *Let \mathfrak{g} be of type $D_n^{(1)}$ and $B = B^{r,1} \otimes B^\bullet$ with $r = n-1, n$. We have*

$$\delta_r = \delta_{\text{sp}}.$$

Proof. When there are no singular strings in $\nu^{(r)}$, it is straightforward to see $\delta_r(\nu, J) = \delta_{\text{sp}}(\nu, J)$ as they both return $u_{\overline{\lambda_r}}$. Next, assume that $\delta_r(\nu, J) = \delta_{\text{sp}}(\nu, J)$ with the same return value $b \in B^{r,1}$. Fix some $a \in I_0$ such that $f_a(b) \neq 0$ and let (ν', J') be such that $\delta_r(\nu', J')$ returns $f_a(b)$ and $\delta_r(\nu', J') = \delta_r(\nu, J)$. Thus, Lemma 3.5 describes the difference between (ν', J') and (ν, J) .

Next we note that $f_a^2(v(b))$ changes $a \mapsto a+1$ and $\overline{a+1} \mapsto \bar{a}$. Suppose a and $\overline{a+1}$ occurs in b at height h and \bar{h} respectively. Since the row added to (ν', J') is larger than any previously selected string, by our assumption, all strings that are selected in (ν, J) by each application of $\tilde{\delta}$ of δ_{sp} must also be selected by those applications in (ν', J') since the addition of the new box does not change the smaller vacancy numbers but makes any singular rows of length at most ℓ in $\nu^{(a \pm 1)}$ non-singular. However, the h and \bar{h} application of δ will now select the added box (doubled by the application of v), but terminate since none of the rows of length at least $2\ell-1$ in $\nu^{(a \pm 1)}$ are singular. Hence $\delta_r(\nu', J') = \delta_{\text{sp}}(\nu', J')$ and return $f_a(b)$.

As every element b can be written as $f_{a_k} \cdots f_{a_1} u_{\overline{\lambda_r}}$ we have $\delta_r = \delta_{\text{sp}}$ by induction. By the recursive definition of Φ , the second claim follows. \square

Next, we consider the case of $D_{n+1}^{(2)}$ and recall that $\widehat{B}^{n,1} = B^{n,1}$ of type $A_{2n-1}^{(1)}$. Let $\widehat{\delta}_n$ denote the map δ_n of type $A_{2n-1}^{(1)}$. Thus, we have the following from the definition of δ_r and Lemma 8.1.

Proposition 8.3. *Let \mathfrak{g} be of type $D_{n+1}^{(2)}$. Then*

$$v \circ \delta_n = \widehat{\delta}_n \circ v.$$

Lemma 8.4. *Let \mathfrak{g} be of type $E_6^{(1)}$ and $B = B^{6,1} \otimes B^\bullet$. Then*

$$\begin{aligned} \delta_6 &= \delta_1 \circ \delta_1 \circ \text{lb}, \\ \delta_1 &= \delta_6 \circ \delta_6 \circ \text{lb}^\vee. \end{aligned}$$

Proof. Similar to the proof of Lemma 8.1. \square

In type $D_{n+1}^{(2)}$, we define $\text{lb}: B^{1,1} \otimes B^\bullet \rightarrow B^{n,1} \otimes B^{n,1} \otimes B^\bullet$ by $\boxed{1} \mapsto 1\bar{n} \otimes n$ and $\boxed{\emptyset} \mapsto \bar{n} \otimes n$.

Lemma 8.5. *Consider $B = B^{\theta,1} \otimes B^\bullet$ of type \mathfrak{g} . Then, we have*

$$\delta_\theta = \begin{cases} \delta_n \circ \delta_1 \circ \text{lb} = \delta_1 \circ \delta_n \circ \text{lb}^\vee & \text{if } \mathfrak{g} = A_n^{(1)}, \\ \delta_1 \circ \delta_1 \circ \text{lb} & \text{if } \mathfrak{g} = D_n^{(1)}, \\ \delta_n \circ \delta_n \circ \text{lb} & \text{if } \mathfrak{g} = D_{n+1}^{(2)}, \\ \delta_1 \circ \delta_1 \circ \text{lb} \circ \delta_1 \circ \text{lb} & \text{if } \mathfrak{g} = E_6^{(1)}, \\ \delta_7 \circ \delta_7 \circ \text{lb} & \text{if } \mathfrak{g} = E_7^{(1)}. \end{cases}$$

Proof. We first consider type $E_6^{(1)}$.

Note that $B(\theta) \subseteq B(\bar{\Lambda}_1)^{\otimes 3}$ is the unique factor with highest weight element $2\bar{6} \otimes \bar{1}6 \otimes 1$. By Lemma 8.4, we have

$$\delta_6 \circ \delta_6 \circ \text{lb} \circ \delta_1 \circ \text{lb} = \delta_6 \circ \delta_1 \circ \text{lb}.$$

Therefore, it is sufficient to show $\delta_\theta = \delta_6 \circ \delta_1 \circ \text{lb}$.

We proceed by induction. The base case is when δ_θ returns u_θ as no boxes are removed, and so we wish to show $\delta_6 \circ \delta_1 \circ \text{lb}$ returns $2\bar{6} \otimes 6$. Adding u_θ under δ_θ^{-1} does not change the rigged configuration. Similarly, adding 6 under δ_6^{-1} only makes all rows in $\nu^{(6)}$ non-singular. Hence, all boxes added by $2\bar{6}$ under δ_1^{-1} must be of length 1, which are all subsequently removed by lb^{-1} . Hence, we have $\delta_\theta = \delta_6 \circ \delta_1 \circ \text{lb}$.

Consider $b' \otimes b \in B(\theta) \subseteq B(\bar{\Lambda}_1) \otimes B(\bar{\Lambda}_6)$. If $f_a(b' \otimes b) = (f_a b') \otimes b$, then the claim follows from induction as the new box is selected on the application of δ_1 . Therefore, suppose $f_a(b' \otimes b) = b' \otimes (f_a b)$. Therefore, we must have $\varepsilon_a(b') = 0 < 1 = \varphi_a(b)$ by the tensor product rule. If $\varphi_a(b') = 1$, then we have $\varphi_a(b' \otimes b) = 2$, and hence, $b' \otimes b$ corresponds to x_{α_a} . In this case, $f_a(b' \otimes b)$ differs by a quasisingular string in $\nu^{(a)}$, hence after $\delta_1 \circ \text{lb}$, this string becomes singular and selected by δ_6 . Thus, the result of $\delta_6 \circ \delta_1 \circ \text{lb}$ is $b' \otimes f_a b$. Therefore, we have $\varphi_a(b') = 0$. Hence, the new singular row in $\nu^{(6)}$ is selected δ_6 . Hence, we have $\delta_\theta = \delta_6 \circ \delta_1 \circ \text{lb}$.

Finally, we consider the case when δ_θ returns \emptyset . In this case, we want to show $\delta_6 \circ \delta_1 \circ \text{lb}$ returns $\bar{1}6$ and $\bar{6}$ respectively. Adding \emptyset under δ_θ^{-1} adds c_a singular rows of length 1 to $\nu^{(a)}$. Next, we note that δ_6^{-1} adding $\bar{1}6$ makes all rows in $\nu^{(6)}$ non-singular. Therefore, adding $\bar{6}$ under δ_1^{-1} can only create singular rows of length 1 and, after performing lb^{-1} , there are precisely c_a such rows. Hence, we have $\delta_\theta = \delta_6 \circ \delta_1 \circ \text{lb}$.

The other cases are similar. \square

Lemma 8.6. *Let \mathfrak{g} be of type $C_n^{(1)}$. Then $\delta_1 = \tilde{\delta}$.*

Proof. This follows from the description of δ_1 and Proposition 7.5. \square

By combining the above results, we have the following.

Theorem 8.7. *Let \mathfrak{g} be of dual untwisted type or type $A_{2n}^{(2)}$, $A_{2n}^{(2)\dagger}$, $C_n^{(1)}$. Let Φ and Φ' be two bijections defined by δ_σ and $\delta_{\sigma'}$ with corresponding lb-diagrams. Then we have*

$$\Phi = \Phi'.$$

Moreover, we have $\Phi = \tilde{\Phi}$.

Conjecture 8.8. *Theorem 8.7 holds for type $B_n^{(1)}$.*

9. THE FILLING MAP

In this section, we characterize all highest weight rigged configurations appearing in $\text{RC}^{HW}(B^{r,s})$ for various $r \in I_0$ in types $E_{6,7,8}^{(1)}$, $E_6^{(2)}$, and $F_4^{(1)}$. Note that it has not been shown that $B^{r,s}$ is the crystal basis of the KR model $W^{r,s}$ in general. For non-exceptional types, this was shown in [OS08] and for certain other special cases in [JS10, KMOY07, Yam98]. We note that when r is a special (resp. adjoint) node, then $W^{r,s}$ has a crystal basis by [OS08, Rem. 3.1] and [KKM⁺92b, Prop. 3.4.4] (resp. [KKM⁺92b, Prop. 3.4.5]). However, this provides further evidence for this to be true as we show many of the graded decompositions agrees with the conjectures of [HKO⁺99, HKO⁺02b]. Using this, we describe the filling map and define *Kirillov–Reshetikhin (KR) tableaux* for $B^{r,s}$ for certain r in types $E_{6,7,8}^{(1)}$ and $E_6^{(2)}$ as an extension of [OSS13, SS15b, Scr16]. We give a table of the KR tableaux for $B^{r,1}$ in Appendix A.

For the nodes we do not consider here, the roots which appear as edge labels in the (ambient) Kleber tree, and hence the weights can appear in the decompositions, are completely determined by those at level 1, i.e., appear in the (ambient) Kleber tree for $B^{r,1}$. Thus, it is possible to determine an explicit parameterization of the classical decomposition of $B^{r,s}$ for all exceptional types, as well as the corresponding rigged configurations (and their cocharge). In particular, this parameterization will be given by integer points in a polytope.

However, the author believes any such parameterization is likely to not be enlightening as it will involve numerous linear inequalities (and possibly some equalities). Then for the filling map, the rules can become even more complicated. For example, consider the parameterization for $B^{2,s}$ of type $D_4^{(3)}$ given in [Scr16, Def. 4.10] and the corresponding filling map.

9.1. Notation. We define some notation to aid in the description of $\text{RC}^{HW}(B^{r,s})$. Consider a tuple $(\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell)})$, where $\alpha^{(k)} = \sum_{a \in I_0} c_a^{(k)} \bar{\alpha}_a \in Q_0^+$ for all $1 \leq k \leq \ell$. We denote by $\nu(\alpha^{(1)}, \alpha^{(2)}, \dots, \alpha^{(\ell)})$ the configuration given by $\nu^{(a)}$ stacking a column of height $c_a^{(k)}$ for all $1 \leq k \leq \ell$ and left justifying this (so it is a partition). We also denote $k * [\alpha]$ as the sequence $(\alpha, \alpha, \dots, \alpha)$ of length k and $k_1 * [\alpha^{(1)}] + k_2 * [\alpha^{(2)}]$ as the concatenation of the two sequences.

9.2. Type $E_6^{(1)}$. For $r = 1$ and $r = 6$ in type $E_6^{(1)}$, we have $B^{r,s} \cong B(s\bar{\Lambda}_r)$ as $U_q(\mathfrak{g}_0)$ -crystals. Thus, the filling map $\text{fill}: B^{r,s} \rightarrow T^{r,s}$ is the identity map. Moreover, we have $\text{RC}(B^{r,s}) \cong \text{RC}(B^{r,s}; s\bar{\Lambda}_r)$ as $U_q(\mathfrak{g}_0)$ -crystals.

Proposition 9.1. *Consider the KR crystal $B^{2,s}$ of type $E_6^{(1)}$. We have*

$$\text{RC}(B^{2,s}) = \bigoplus_{k=0}^s \text{RC}(B^{2,s}; (s-k)\bar{\Lambda}_2).$$

Moreover, the highest weight rigged configurations in $\text{RC}(B^{2,s})$ are given by $\nu(k * [\alpha])$ with all riggings 0, where

$$\alpha = \bar{\alpha}_1 + 2\bar{\alpha}_2 + 2\bar{\alpha}_3 + 3\bar{\alpha}_4 + 2\bar{\alpha}_5 + \bar{\alpha}_6 = \bar{\Lambda}_2,$$

and

$$\text{cc}(\nu, J) = k.$$

Proof. Note that by Condition (K2) of Definition 2.15, the only root that we can subtract from $\bar{\Lambda}_2$ is α . Thus the Kleber tree $T(B^{2,s})$ is a path where the node at depth k has weight $\bar{\Lambda}_2 - k\alpha$ for $0 \leq k \leq s$. Hence, the rigged configuration is given by $\nu(k * [\alpha])$. It is straightforward to check that $\text{cc}(\nu, J) = k$. \square

Definition 9.2. We define the filling map $\text{fill}: B^{2,s} \rightarrow T^{2,s}$ on the classically highest weight element $b \in B((s-k)\bar{\Lambda}_2) \subseteq B^{2,s}$ by defining $\text{fill}(b)$ as the tableau with the first $s-k$ columns as $[1, \bar{1}6, 2\bar{6}]$, the next $\lfloor k/2 \rfloor$ columns as

$\bar{2}5$	1
$\bar{5}6$	$\bar{1}6$
$\bar{6}$	$2\bar{6}$

and if k is odd, the final column as $[1, \bar{1}6, \bar{6}]$. We then extend fill as a classical crystal isomorphism.

Example 9.3. Consider $B^{2,8}$ of type $E_6^{(1)}$. Then $B(k\bar{\Lambda}_2) \subseteq B^{2,8}$ corresponds to the classically highest weight element

$$\begin{aligned}
k=4 \rightsquigarrow & \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & 1 & \bar{2}5 & 1 & \bar{2}5 & 1 \\ \hline \bar{1}6 & \bar{1}6 & \bar{1}6 & \bar{1}6 & \bar{5}6 & \bar{1}6 & \bar{5}6 & \bar{1}6 \\ \hline 2\bar{6} & 2\bar{6} & 2\bar{6} & 2\bar{6} & \bar{6} & 2\bar{6} & \bar{6} & 2\bar{6} \\ \hline \end{array}, \\
k=5 \rightsquigarrow & \begin{array}{|c|c|c|c|c|c|c|c|} \hline 1 & 1 & 1 & \bar{2}5 & 1 & \bar{2}5 & 1 & 1 \\ \hline \bar{1}6 & \bar{1}6 & \bar{1}6 & \bar{5}6 & \bar{1}6 & \bar{5}6 & \bar{1}6 & \bar{1}6 \\ \hline 2\bar{6} & 2\bar{6} & 2\bar{6} & \bar{6} & 2\bar{6} & \bar{6} & 2\bar{6} & \bar{6} \\ \hline \end{array},
\end{aligned}$$

where the shaded regions are the parts that are “filled in.”

Proposition 9.4. *Let $\text{fill}: B^{2,s} \rightarrow T^{2,s}$ given by Definition 9.2 and ι be the natural (classical) crystal isomorphism. We have*

$$\Phi = \text{fill} \circ \iota$$

on classically highest weight elements.

Proof. If we apply column splitting ls to $B^{2,s}$ when $s > k$, then we make all rows in $\nu^{(2)}$ nonsingular. A straightforward computation shows that we obtain $[1, \overline{13}, 2\overline{3}]$ and the rigged configuration has not changed. If $s = k$, then ls keeps all rows of length k being singular. It is a straightforward computation shows the column is $[1\overline{2}, \overline{16}, 6]$ when $k > 1$ and the new rigged configuration is obtained by deleting two boxes from every row of $\nu^{(a)}$ for all $a \in I_0$. If $k = 1$, then a finite computation shows we obtain $[1, \overline{16}, \overline{6}]$. \square

Proposition 9.5. *Consider the KR crystal $B^{3,s}$ of type $E_6^{(1)}$. We have*

$$\text{RC}(B^{3,s}) = \bigoplus_{k=0}^s \text{RC}(B^{3,s}; (s-k)\overline{\Lambda}_3 + k\overline{\Lambda}_6).$$

Moreover the highest weight rigged configurations in $\text{RC}(B^{3,s})$ are given by $\nu(k * [\alpha])$ with all riggings 0, where

$$\alpha = \overline{\alpha}_1 + \overline{\alpha}_2 + 2\overline{\alpha}_3 + 2\overline{\alpha}_4 + \overline{\alpha}_5 = \overline{\Lambda}_3 - \overline{\Lambda}_6,$$

and

$$\text{cc}(\nu, J) = k.$$

Proof. Similar to the proof of Proposition 9.1. \square

Definition 9.6. We define the filling map $\text{fill}: B^{3,s} \rightarrow T^{3,s}$ on classically highest weight element $b \in B((s-k)\overline{\Lambda}_3 + k\overline{\Lambda}_6) \subseteq B^{3,s}$ by defining $\text{fill}(b)$ as the tableau with the first $s-k$ columns as $[1, \overline{13}]$, the next $\lfloor k/2 \rfloor$ columns as

$1\overline{3}6$	1
$\overline{1}6$	$\overline{1}3$

and if k is odd, the last column as $[1, \overline{16}]$. We then extend fill as a classical crystal isomorphism.

Proposition 9.7. *Let $\text{fill}: B^{3,s} \rightarrow T^{3,s}$ given by Definition 9.6 and ι be the natural (classical) crystal isomorphism. We have*

$$\Phi = \text{fill} \circ \iota$$

on classically highest weight elements.

Proof. Similar to the proof of Proposition 9.4. \square

The case for $r = 5$ is dual to the above. In particular, the proofs of the following propositions are similar to those for the $r = 3$ case.

Proposition 9.8. *Consider the KR crystal $B^{5,s}$ of type $E_6^{(1)}$. We have*

$$\text{RC}(B^{5,s}) = \bigoplus_{k=0}^s \text{RC}(B^{5,s}; (s-k)\overline{\Lambda}_5 + k\overline{\Lambda}_1).$$

Moreover the highest weight rigged configurations in $\text{RC}(B^{5,s})$ are given by $\nu(k * [\alpha])$ with all riggings 0, where

$$\alpha = \overline{\alpha}_2 + \overline{\alpha}_3 + 2\overline{\alpha}_4 + \overline{\alpha}_5 + \overline{\alpha}_6 = \overline{\Lambda}_5 - \overline{\Lambda}_1,$$

and

$$\text{cc}(\nu, J) = k.$$

Definition 9.9. We define the filling map $\text{fill}: B^{5,s} \rightarrow T^{5,s}$ on classically highest weight element $b \in B((s-k)\overline{\Lambda}_5 + k\overline{\Lambda}_1) \subseteq B^{5,s}$ by defining $\text{fill}(b)$ as the tableau with the first $s-k$ columns be $[1, \overline{13}, 2\overline{3}, \overline{25}]$, the next $\lfloor k/2 \rfloor$ columns as

1	1
$2\overline{5}6$	$\overline{1}3$
$\overline{6}$	$2\overline{3}$
$1\overline{2}$	$\overline{2}5$

and if k is odd, the last column as $[1, \overline{13}, 2\overline{3}, \overline{21}]$. We then extend fill as a classical crystal isomorphism.

Proposition 9.10. Let $\text{fill}: B^{5,s} \rightarrow T^{5,s}$ given by Definition 9.9 and ι be the natural (classical) crystal isomorphism. We have

$$\Phi = \text{fill} \circ \iota$$

on classically highest weight elements.

Proposition 9.11. Consider the KR crystal $B^{4,s}$ of type $E_6^{(1)}$. We have

$$\text{RC}(B^{4,s}) = \bigoplus_{\lambda} \text{RC}(B^{4,s}; \lambda)^{\oplus(1+k_2-k_4-2k_5)},$$

where $\lambda = s\bar{\Lambda}_4 - \sum_{i=1}^5 k_i \bar{\alpha}^{(i)}$ with

$$\begin{aligned} \bar{\alpha}^{(1)} &:= 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 4\bar{\alpha}_3 + 6\bar{\alpha}_4 + 4\bar{\alpha}_5 + 2\bar{\alpha}_6 &= \bar{\Lambda}_4, \\ \bar{\alpha}^{(2)} &:= \bar{\alpha}_1 + \bar{\alpha}_2 + 2\bar{\alpha}_3 + 3\bar{\alpha}_4 + 2\bar{\alpha}_5 + \bar{\alpha}_6 &= \bar{\Lambda}_4 - \bar{\Lambda}_2, \\ \bar{\alpha}^{(3)} &:= \bar{\alpha}_2 + \bar{\alpha}_3 + 2\bar{\alpha}_4 + \bar{\alpha}_5 &= \bar{\Lambda}_4 - \bar{\Lambda}_1 - \bar{\Lambda}_6, \\ \bar{\alpha}^{(4)} &:= \bar{\alpha}_2 + \bar{\alpha}_4 &= \bar{\Lambda}_2 + \bar{\Lambda}_4 - (\bar{\Lambda}_3 + \bar{\Lambda}_5), \\ \bar{\alpha}^{(5)} &:= \bar{\alpha}_2 &= 2\bar{\Lambda}_2 - \bar{\Lambda}_4, \end{aligned}$$

such that

- (1) $k_1 + k_2 + k_3 + k_4 \leq s$,
- (2) $k_i \geq 0$ for $i = 1, 2, 3, 4, 5$, and
- (3) $k_2 \geq k_4 + 2k_5$.

Moreover the highest weight rigged configurations in $\text{RC}(B^{4,s})$ are given by

$$\nu(k_1 * [\bar{\alpha}^{(1)}] + k_2 * [\bar{\alpha}^{(2)}] + k_3 * [\bar{\alpha}^{(3)}] + k_4 * [\bar{\alpha}^{(4)}] + k_5 * [\bar{\alpha}^{(5)}]),$$

with all riggings 0 except for x in the first row of $\nu^{(2)}$, which satisfies $0 \leq x \leq k_2 - k_4 - 2k_5$, and

$$\text{cc}(\nu, J) = 3k_1 + k_2 + k_3 + k_4 + k_5 + x.$$

Proof. Note that $\bar{\alpha}^{(1)} > \bar{\alpha}^{(2)} > \bar{\alpha}^{(3)} > \bar{\alpha}^{(4)} > \bar{\alpha}^{(5)}$ component-wise expressed in terms of $\{\bar{\alpha}_i\}_{i \in I_0}$. The claim for the rigged configurations follows from Definition 2.15 and the description of $\bar{\alpha}^{(i)}$ for $i = 1, 2, 3, 4, 5$. The claim for the cocharge is a straightforward computation. \square

We first note that for $r = 1, 2, 3, 5, 6$, it is clear that these graded decompositions agree with the conjectures in [HKO⁺99, App. A] (with some straightforward relabeling). Thus, we show the $r = 4$ case.

Proposition 9.12. Let $q^k B(\bar{\Lambda}_r)$ denote that $B(\bar{\Lambda}_r)$ is given a grading of k . We have

$$\begin{aligned} B^{4,s} &\cong \bigoplus_{\substack{j_1+j_2+2j_3+j_4 \leq s \\ j_1, j_2, j_3, j_4 \in \mathbb{Z}_{\geq 0}}} \min(1+j_2, 1+s-j_1-j_2-2j_3-j_4) q^{3s-2j_1-3j_2-4j_3-2j_4} \\ &\quad \times \sum_{k=0}^{j_1} q^k B(j_1 \bar{\Lambda}_2 + j_2 \bar{\Lambda}_4 + j_3 (\bar{\Lambda}_3 + \bar{\Lambda}_5) + j_4 (\bar{\Lambda}_1 + \bar{\Lambda}_6)). \end{aligned}$$

Proof. First note that

$$\begin{aligned} \text{wt}(\nu, J) &= s\bar{\Lambda}_4 - k_1 \bar{\Lambda}_2 - k_2 (\bar{\Lambda}_4 - \bar{\Lambda}_2) - k_3 (\bar{\Lambda}_4 - \bar{\Lambda}_1 - \bar{\Lambda}_6) \\ &\quad - k_4 (\bar{\Lambda}_2 + \bar{\Lambda}_4 - (\bar{\Lambda}_3 + \bar{\Lambda}_5)) - k_5 (2\bar{\Lambda}_2 - \bar{\Lambda}_4) \\ &= (k_2 - k_4 - 2k_5) \bar{\Lambda}_2 + (s - k_1 - k_2 - k_3 - k_4 + k_5) \bar{\Lambda}_4 + k_4 (\bar{\Lambda}_3 + \bar{\Lambda}_5) + k_3 (\bar{\Lambda}_1 + \bar{\Lambda}_6) \end{aligned}$$

Hence, we must have

$$\begin{aligned} j_1 &= k_2 - k_4 - 2k_5, & j_2 &= s - k_1 - k_2 - k_3 - k_4 + k_5, \\ j_3 &= k_4, & j_4 &= k_3. \end{aligned} \tag{9.1}$$

Note that the conditions on k_1, k_2, k_3, k_4, k_5 guarantee that $j_1, j_2, j_3, j_4 \geq 0$. Next, we have

$$\begin{aligned} j_1 + j_2 + 2j_3 + j_4 &= s - k_1 - k_5 \leq s, \\ 3s - 2j_1 - 3j_2 - 4j_3 - 2j_4 &= 3k_1 + k_2 + k_3 + k_4 + k_5 = \text{cc}(\nu). \end{aligned}$$

Additionally, we want the multiplicity to agree, so we need to show

$$M := \min(1 + j_2, 1 + s - j_1 - j_2 - 2j_3 - j_4)$$

equals the number of time a node of weight $j_1\bar{\Lambda}_2 + j_2\bar{\Lambda}_4 + j_3(\bar{\Lambda}_3 + \bar{\Lambda}_5) + j_4(\bar{\Lambda}_1 + \bar{\Lambda}_6)$ occurs since the multiplicity of a node equals $1 + k_2 - k_4 - 2k_5 = 1 + j_1$. We note that $\bar{\alpha}^{(1)} = 2\bar{\alpha}^{(2)} + \bar{\alpha}^{(5)}$. Explicitly, we fix some $j_1, j_2, j_3, j_4 \in \mathbb{Z}_{\geq 0}$ such that $j_1 + j_2 + 2j_3 + j_4 \leq s$, and we want to show M equals the number of $k_1, k_2, k_3, k_4, k_5 \in \mathbb{Z}_{\geq 0}$ such that

$$s \geq k_1 + k_2 + k_3 + k_4, \quad (9.2a)$$

$$k_2 \geq k_4 + 2k_5, \quad (9.2b)$$

and, from Equation (9.1), that:

$$\begin{aligned} k_1 + k_5 &= s - j_1 - j_2 - 2j_3 - j_4, & k_2 - 2k_5 &= j_1 + j_3, \\ k_3 &= j_4, & k_4 &= j_3. \end{aligned} \quad (9.3)$$

Immediately, we have $k_3, k_4 \geq 0$ and are completely determined since we have fixed $j_3, j_4 \in \mathbb{Z}_{\geq 0}$. We note that

$$k_2 = j_1 + j_3 + 2k_5 \geq j_3 + 2k_5 = k_4 + 2k_5, \quad (9.4)$$

$$k_1 + k_2 + k_3 + k_4 = s - j_2 + k_5, \quad (9.5)$$

and so if we take $k_5 = 0$, then k_1 and k_2 are completely determined and give a valid set. Next, since we want to look over all possible values that give rise to the same weight, the linear dependence $\bar{\alpha}^{(1)} = 2\bar{\alpha}^{(2)} + \bar{\alpha}^{(5)}$ implies we can replace $k_1 \mapsto 2k_2 + k_5$. Note Equation (9.4), and hence Equation (9.2b), still holds under this replacement, but from Equation (9.5) and Equation (9.2a), we have $k_5 \leq j_2$ (so we can do this replacement at most j_2 times). However, we can only do this at most $s - j_1 - j_2 - 2j_3 - j_4$ times since it is the maximum value of k_1 . Thus, we have exactly M possible values for k_1, k_2, k_3, k_4 , and k_5 . \square

9.3. Type $E_7^{(1)}$. For $r = 7$ in type $E_7^{(1)}$, we have $\text{RC}(B^{7,s}) \cong \text{RC}(B^{7,s}; s\bar{\Lambda}_7)$. Therefore the filling map $\text{fill}: B^{7,s} \rightarrow T^{7,s}$ is the identity map.

Proposition 9.13. *Consider the KR crystal $B^{1,s}$ of type $E_7^{(1)}$. We have*

$$\text{RC}(B^{1,s}) \cong \bigoplus_{k=0}^s \text{RC}(B^{1,s}; k\bar{\Lambda}_1).$$

Moreover the highest weight rigged configurations in $\text{RC}(B^{3,s})$ are given by $\nu(k * [\alpha])$ with all riggings 0, where

$$\alpha = 2\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3 + 4\bar{\alpha}_4 + 3\bar{\alpha}_5 + 2\bar{\alpha}_6 + \bar{\alpha}_7 = \bar{\Lambda}_1,$$

and

$$\text{cc}(\nu, J) = k.$$

Proof. Similar to the proof of Proposition 9.1. \square

Definition 9.14. We define the filling map $\text{fill}: B^{1,s} \rightarrow T^{1,s}$ on classically highest weight element $b \in B((s-k)\bar{\Lambda}_1) \subseteq B^{1,s}$ by defining $\text{fill}(b)$ as the tableau with the first $s-k$ columns as $[7, 1\bar{7}]$, the next $\lfloor k/2 \rfloor$ columns as

$\bar{1}7$	7
$\bar{7}$	$1\bar{7}$

and if k is odd, the last column as $[7, \bar{7}]$. We then extend fill as a classical crystal isomorphism.

Proposition 9.15. *Let $\text{fill}: B^{1,s} \rightarrow T^{1,s}$ given by Definition 9.14 and ι be the natural (classical) crystal isomorphism. We have*

$$\Phi = \text{fill} \circ \iota$$

on classically highest weight elements.

Proof. Similar to the proof of Proposition 9.4. \square

Proposition 9.16. Consider the KR crystal $B^{2,s}$ of type $E_7^{(1)}$. We have

$$\mathrm{RC}(B^{2,s}) \cong \bigoplus_{k=0}^s \mathrm{RC}(B^{2,s}; (s-k)\overline{\Lambda}_2 + k\overline{\Lambda}_7)$$

Moreover the highest weight rigged configurations in $\mathrm{RC}(B^{3,s})$ are given by $\nu(k * [\alpha])$ with all riggings 0, where

$$\alpha = \overline{\alpha}_1 + 2\overline{\alpha}_2 + 2\overline{\alpha}_3 + 3\overline{\alpha}_4 + 2\overline{\alpha}_5 + \overline{\alpha}_6 = \overline{\Lambda}_2 - \overline{\Lambda}_7,$$

and

$$\mathrm{cc}(\nu, J) = k.$$

Proof. Similar to the proof of Proposition 9.1. □

Definition 9.17. We define the filling map $\mathrm{fill}: B^{2,s} \rightarrow T^{2,s}$ on classically highest weight element $b \in B((s-k)\overline{\Lambda}_2 + k\overline{\Lambda}_7) \subseteq B^{2,s}$ by defining $\mathrm{fill}(b)$ as the tableau with the first $s-k$ columns as $[7, 1\overline{7}, \overline{1}2]$, the next $\lfloor k/2 \rfloor$ columns as

7	7
$1\overline{2}$	$1\overline{7}$
$\overline{1}7$	$\overline{1}2$

and if k is odd, the last column as $[7, 1\overline{7}, \overline{1}7]$. We then extend fill as a classical crystal isomorphism.

Proposition 9.18. Let $\mathrm{fill}: B^{2,s} \rightarrow T^{2,s}$ given by Definition 9.17 and ι be the natural (classical) crystal isomorphism. We have

$$\Phi = \mathrm{fill} \circ \iota$$

on classically highest weight elements.

Proof. Similar to the proof of Proposition 9.4. □

Proposition 9.19. Consider the KR crystal $B^{6,s}$ of type $E_7^{(1)}$. We have

$$\mathrm{RC}(B^{6,s}) \cong \bigoplus_{\substack{k_1+k_2 \leq s \\ k_1, k_2 \in \mathbb{Z}_{\geq 0}}} \mathrm{RC}(B^{6,s}; (s-k_1-k_2)\overline{\Lambda}_6 + k_2\overline{\Lambda}_1).$$

Moreover the highest weight rigged configurations in $\mathrm{RC}(B^{6,s})$ are given by

$$\nu(k_1 * [\overline{\alpha}^{(1)}] + k_2 * [\overline{\alpha}^{(2)}]),$$

where

$$\begin{aligned} \overline{\alpha}^{(1)} &= 2\overline{\alpha}_1 + 3\overline{\alpha}_2 + 4\overline{\alpha}_3 + 6\overline{\alpha}_4 + 5\overline{\alpha}_5 + 4\overline{\alpha}_6 + 2\overline{\alpha}_7 &= \overline{\Lambda}_6, \\ \overline{\alpha}^{(2)} &= \overline{\alpha}_2 + \overline{\alpha}_3 + 2\overline{\alpha}_4 + 2\overline{\alpha}_5 + 2\overline{\alpha}_6 + \overline{\alpha}_7 &= \overline{\Lambda}_6 - \overline{\Lambda}_1, \end{aligned}$$

with all riggings 0 and

$$\mathrm{cc}(\nu, J) = 2k_1 + k_2.$$

Proof. Similar to the proof of Proposition 9.11 as $\overline{\alpha}^{(1)} > \overline{\alpha}^{(2)}$. □

Proposition 9.20. Consider the KR crystal $B^{3,s}$ of type $E_7^{(1)}$. We have

$$\mathrm{RC}(B^{3,s}) = \bigoplus_{\lambda} \mathrm{RC}(B^{3,s}; \lambda)^{\oplus (1+k_2-k_4-2k_5)},$$

where $\lambda = s\bar{\Lambda}_3 - \sum_{i=1}^5 k_i \bar{\alpha}^{(i)}$ with

$$\begin{aligned}\bar{\alpha}^{(1)} &= 3\bar{\alpha}_1 + 4\bar{\alpha}_2 + 6\bar{\alpha}_3 + 8\bar{\alpha}_4 + 6\bar{\alpha}_5 + 4\bar{\alpha}_6 + 2\bar{\alpha}_7 &= \bar{\Lambda}_3 \\ \bar{\alpha}^{(2)} &= \bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3 + 4\bar{\alpha}_4 + 3\bar{\alpha}_5 + 2\bar{\alpha}_6 + 1\bar{\alpha}_7 &= \bar{\Lambda}_3 - \bar{\Lambda}_1 \\ \bar{\alpha}^{(3)} &= \bar{\alpha}_1 + \bar{\alpha}_2 + 2\bar{\alpha}_3 + 2\bar{\alpha}_4 + \bar{\alpha}_5 &= \bar{\Lambda}_3 - \bar{\Lambda}_6 \\ \bar{\alpha}^{(4)} &= \bar{\alpha}_1 + \bar{\alpha}_3 &= \bar{\Lambda}_1 + \bar{\Lambda}_3 - \bar{\Lambda}_4 \\ \bar{\alpha}^{(5)} &= \bar{\alpha}_1 &= 2\bar{\Lambda}_1 - \bar{\Lambda}_3\end{aligned}$$

such that

- (1) $k_1 + k_2 + k_3 + k_4 \leq s$,
- (2) $k_i \geq 0$ for $i = 1, 2, 3, 4, 5$, and
- (3) $k_2 \geq k_4 + 2k_5$.

Moreover the highest weight rigged configurations in $\text{RC}(B^{3,s})$ are given by

$$\nu(k_1 * [\bar{\alpha}^{(1)}] + k_2 * [\bar{\alpha}^{(2)}] + k_3 * [\bar{\alpha}^{(3)}] + k_4 * [\bar{\alpha}^{(4)}] + k_5 * [\bar{\alpha}^{(5)}]),$$

with all riggings 0 except for x in the first row of $\nu^{(1)}$, which satisfies $0 \leq x \leq k_2 - k_4 - 2k_5$, and

$$\text{cc}(\nu, J) = 3k_1 + k_2 + k_3 + k_4 + k_5 + x.$$

Proof. Similar to the proof of Proposition 9.11. □

We note that our graded decompositions for $r = 1, 2, 3, 6, 7$ agree with the conjectures of [HKO⁺99, App. A], where the case of $r = 3$ is similar to the proof of Proposition 9.12.

9.4. Type $E_8^{(1)}$. For type $E_8^{(1)}$, we do not have any minuscule nodes as there are no diagram automorphism. Thus, we begin with the adjoint node.

Proposition 9.21. *Consider the KR crystal $B^{8,s}$ of type $E_8^{(1)}$. We have*

$$\text{RC}(B^{8,s}) = \bigoplus_{k=0}^s \text{RC}(B^{8,s}; (s-k)\bar{\Lambda}_8).$$

Moreover the highest weight rigged configurations in $\text{RC}(B^{8,s})$ are given by $\nu(k * [\alpha])$ with all riggings 0, where

$$\alpha = 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 4\bar{\alpha}_3 + 6\bar{\alpha}_4 + 5\bar{\alpha}_5 + 4\bar{\alpha}_6 + 3\bar{\alpha}_7 + 2\bar{\alpha}_8 = \bar{\Lambda}_8$$

and $\text{cc}(\nu, J) = k$.

Proof. The proof is similar to Proposition 9.1. □

Definition 9.22. We define the filling map $\text{fill}: B^{8,s} \rightarrow T^{8,s}$ on classically highest weight element $b \in B((s-k)\bar{\Lambda}_8) \subseteq B^{8,s}$ by defining $\text{fill}(b)$ as the tableau with the first $s-k$ columns as $[8]$, the next $\lfloor k/2 \rfloor$ columns as $\begin{bmatrix} 8 \\ 8 \end{bmatrix}$, and if k is odd, the last column as $[\emptyset]$. We then extend fill as a classical crystal isomorphism.

Proposition 9.23. *Let $\text{fill}: B^{8,s} \rightarrow T^{8,s}$ given by Definition 9.22 and ι be the natural (classical) crystal isomorphism. We have*

$$\Phi = \text{fill} \circ \iota$$

on classically highest weight elements.

Proof. Similar to the proof of Proposition 9.4. □

Proposition 9.24. *Consider the KR crystal $B^{7,s}$ of type $E_8^{(1)}$. We have*

$$\text{RC}(B^{7,s}) = \bigoplus_{\lambda} \text{RC}(B^{7,s}; \lambda)^{\oplus (1+k_2-k_4-2k_5)}$$

where

$$\begin{aligned}
\bar{\alpha}^{(1)} &= 4\bar{\alpha}_1 + 5\bar{\alpha}_2 + 8\bar{\alpha}_3 + 12\bar{\alpha}_4 + 10\bar{\alpha}_5 + 8\bar{\alpha}_6 + 6\bar{\alpha}_7 + 3\bar{\alpha}_8 &= \bar{\Lambda}_7, \\
\bar{\alpha}^{(2)} &= 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 4\bar{\alpha}_3 + 6\bar{\alpha}_4 + 5\bar{\alpha}_5 + 4\bar{\alpha}_6 + 3\bar{\alpha}_7 + \bar{\alpha}_8 &= \bar{\Lambda}_7 - \bar{\Lambda}_8, \\
\bar{\alpha}^{(3)} &= \bar{\alpha}_2 + \bar{\alpha}_3 + 2\bar{\alpha}_4 + 2\bar{\alpha}_5 + 2\bar{\alpha}_6 + 2\bar{\alpha}_7 + \bar{\alpha}_8 &= \bar{\Lambda}_7 - \bar{\Lambda}_1, \\
\bar{\alpha}^{(4)} &= \bar{\alpha}_7 + \bar{\alpha}_8 &= \bar{\Lambda}_7 + \bar{\Lambda}_8 - \bar{\Lambda}_6, \\
\bar{\alpha}^{(5)} &= \bar{\alpha}_8 &= 2\bar{\Lambda}_8 - \bar{\Lambda}_7,
\end{aligned}$$

and $\lambda = s\bar{\Lambda}_7 - \sum_{i=1}^5 k_i \bar{\alpha}^{(i)}$, such that

- (1) $k_1 + k_2 + k_3 + k_4 \leq s$,
- (2) $k_i \geq 0$ for $i = 1, 2, 3, 4, 5$, and
- (3) $k_2 \geq k_4 + 2k_5$.

Moreover the highest weight rigged configurations in $\text{RC}(B^{8,s})$ are given by

$$\nu(k_1 * [\bar{\alpha}^{(1)}] + k_2 * [\bar{\alpha}^{(2)}] + k_3 * [\bar{\alpha}^{(3)}] + k_4 * [\bar{\alpha}^{(4)}] + k_5 * [\bar{\alpha}^{(5)}])$$

with all riggings 0 except the first row of $\nu^{(8)}$, which satisfies $0 \leq x \leq k_2 - k_4 - 2k_5$, and

$$\text{cc}(\nu, J) = 3k_1 + k_2 + k_3 + k_4 + k_5 + x.$$

Proof. Similar to the proof of Proposition 9.11. □

Proposition 9.25. Consider the KR crystal $B^{1,s}$ of type $E_8^{(1)}$. We have

$$\text{RC}(B^{1,s}) = \bigoplus_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s}} \text{RC}(B^{1,s}; (s - k_1 - k_2)\bar{\Lambda}_1 + k_2\bar{\Lambda}_8).$$

Moreover the highest weight rigged configurations in $\text{RC}(B^{1,s})$ are given by

$$\nu(k_1 * [\bar{\alpha}^{(1)}] + k_2 * [\bar{\alpha}^{(2)}]),$$

where

$$\begin{aligned}
\bar{\alpha}^{(1)} &= 4\bar{\alpha}_1 + 5\bar{\alpha}_2 + 7\bar{\alpha}_3 + 10\bar{\alpha}_4 + 8\bar{\alpha}_5 + 6\bar{\alpha}_6 + 4\bar{\alpha}_7 + 2\bar{\alpha}_8 &= \bar{\Lambda}_1, \\
\bar{\alpha}^{(2)} &= 2\bar{\alpha}_1 + 2\bar{\alpha}_2 + 3\bar{\alpha}_3 + 4\bar{\alpha}_4 + 3\bar{\alpha}_5 + 2\bar{\alpha}_6 + \bar{\alpha}_7 &= \bar{\Lambda}_1 - \bar{\Lambda}_8,
\end{aligned}$$

with all riggings 0 and

$$\text{cc}(\nu, J) = 2k_1 + k_2.$$

Proof. Similar to the proof of Proposition 9.11 as $\bar{\alpha}^{(1)} > \bar{\alpha}^{(2)}$. □

We note that our graded decompositions for $r = 1, 7, 8$ agree with the conjectures of [HKO⁺99, App. A], where the case of $r = 7$ is similar to the proof of Proposition 9.12.

Remark 9.26. The cases of $r = 4$ of type $E_6^{(1)}$, $r = 3$ of type $E_7^{(1)}$, and $r = 7$ of type $E_8^{(1)}$ are all nodes which are distance 2 from the affine node, and all have the same graded decompositions. Thus it would be interesting to see if there is a uniform describe of these KR crystals, and additionally to compare it with $B^{r,s}$ for $r = 1, 3$ in type $D_n^{(1)}$ and $r = 2, n - 2$ in type $A_n^{(1)}$.

9.5. Type $E_6^{(2)}$. For $r = 1$, this is immediate deduced from the devirtualization of Proposition 9.1 ($r = 2$ in type $E_6^{(1)}$) as $\gamma_a = 1$ for all $a \in I$.

Proposition 9.27. Consider the KR crystal $B^{1,s}$ of type $E_6^{(2)}$. We have

$$\text{RC}(B^{1,s}) = \bigoplus_{k=0}^s \text{RC}(B^{1,s}; (s - k)\bar{\Lambda}_1).$$

Moreover, the highest weight rigged configurations in $\text{RC}(B^{1,s})$ are given by $\nu(k * [\alpha])$ with all riggings 0, where

$$\alpha = 2\bar{\alpha}_1 + 3\bar{\alpha}_2 + 2\bar{\alpha}_3 + \bar{\alpha}_4 = \bar{\Lambda}_1,$$

and

$$\text{cc}(\nu, J) = k.$$

Proof. The proof is similar to Proposition 9.1 as the ambient Kleber tree is the same as the virtual Kleber tree. \square

Definition 9.28. We define the filling map $\text{fill}: B^{1,s} \rightarrow T^{1,s}$ on classically highest weight element $b \in B((s-k)\overline{\Lambda}_1) \subseteq B^{1,s}$ by defining $\text{fill}(b)$ as the tableau with the first $s-k$ columns as $[1]$, the next $\lfloor k/2 \rfloor$ columns as $\begin{bmatrix} 1 & 1 \end{bmatrix}$, and if k is odd, the last column as $[\emptyset]$. We then extend fill as a classical crystal isomorphism.

Proposition 9.29. Let $\text{fill}: B^{1,s} \rightarrow T^{1,s}$ given by Definition 9.28 and ι be the natural (classical) crystal isomorphism. We have

$$\Phi = \text{fill} \circ \iota$$

on classically highest weight elements.

Proof. Similar to the proof of Proposition 9.4. \square

Proposition 9.30. Consider the KR crystal $B^{2,s}$ of type $E_6^{(2)}$. We have

$$\text{RC}(B^{2,s}) = \bigoplus_{\lambda} \text{RC}(B^{2,s}; \lambda)^{\oplus(1+k_2-k_4-2k_5)},$$

where

$$\begin{aligned} \overline{\alpha}^{(1)} &:= 3\overline{\alpha}_1 + 6\overline{\alpha}_2 + 4\overline{\alpha}_3 + 2\overline{\alpha}_4 &= \overline{\Lambda}_2, \\ \overline{\alpha}^{(2)} &:= \overline{\alpha}_1 + 3\overline{\alpha}_2 + 2\overline{\alpha}_3 + \overline{\alpha}_4 &= \overline{\Lambda}_2 - \overline{\Lambda}_1, \\ \overline{\alpha}^{(3)} &:= \overline{\alpha}_1 + 2\overline{\alpha}_2 + \overline{\alpha}_3 &= \overline{\Lambda}_2 - \overline{\Lambda}_4, \\ \overline{\alpha}^{(4)} &:= \overline{\alpha}_1 + \overline{\alpha}_2 &= \overline{\Lambda}_1 + \overline{\Lambda}_2 - \overline{\Lambda}_3, \\ \overline{\alpha}^{(5)} &:= \overline{\alpha}_1 &= 2\overline{\Lambda}_1 - \overline{\Lambda}_2, \end{aligned}$$

and $\lambda = s\overline{\Lambda}_2 - \sum_{i=1}^5 k_i \overline{\alpha}^{(i)}$, such that

- (1) $k_1 + k_2 + k_3 + k_4 \leq s$,
- (2) $k_i \geq 0$ for $i = 1, 2, 3, 4, 5$, and
- (3) $k_2 \geq k_4 + 2k_5$.

Moreover the highest weight rigged configurations in $\text{RC}(B^{2,s})$ are given by

$$\nu(k_1 * [\overline{\alpha}^{(1)}] + k_2 * [\overline{\alpha}^{(2)}] + k_3 * [\overline{\alpha}^{(3)}] + k_4 * [\overline{\alpha}^{(4)}] + k_5 * [\overline{\alpha}^{(5)}]),$$

with all riggings 0 except for x in the first row of $\nu^{(1)}$, which satisfies $0 \leq x \leq k_2 - k_4 - 2k_5$, and

$$\text{cc}(\nu, J) = 3k_1 + k_2 + k_3 + k_4 + k_5 + x.$$

Proof. Note that $\overline{\alpha}^{(1)} > \overline{\alpha}^{(2)} > \overline{\alpha}^{(3)} > \overline{\alpha}^{(4)} > \overline{\alpha}^{(5)}$ component-wise expressed in terms of $\{\overline{\alpha}_i\}_{i \in I_0}$. Thus any sequence $(k_1, k_2, k_3, k_4, k_5)$ uniquely determines a path in the ambient Kleber tree, which is the same tree constructed in Proposition 9.11. Hence the claim follows from Definition 2.17 as all $\widehat{\alpha}^{(i)}$ are symmetric with respect to the diagram automorphism ϕ and that $\gamma_a = 1$ for all $a \in I$. \square

Proposition 9.31. Consider the KR crystal $B^{4,s}$ of type $E_6^{(2)}$. We have

$$\text{RC}(B^{4,s}) = \bigoplus_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s}} \text{RC}(B^{4,s}; k_2 \overline{\Lambda}_1 + (s - k_1 - k_2) \overline{\Lambda}_4).$$

Moreover, the highest weight rigged configurations in $\text{RC}(B^{4,s})$ are given by

$$\nu(k_1 * [\overline{\alpha}^{(1)}] + k_2 * [\overline{\alpha}^{(2)}]),$$

where

$$\begin{aligned} \overline{\alpha}^{(1)} &= 2\overline{\alpha}_1 + 4\overline{\alpha}_2 + 3\overline{\alpha}_3 + 2\overline{\alpha}_4 &= \overline{\Lambda}_4, \\ \overline{\alpha}^{(2)} &= \overline{\alpha}_2 + \overline{\alpha}_3 + \overline{\alpha}_4 &= \overline{\Lambda}_4 - \overline{\Lambda}_1, \end{aligned}$$

with all riggings 0 and

$$\text{cc}(\nu, J) = 2k_1 + k_2.$$

Proof. We note that $\overline{\alpha}^{(1)} > \overline{\alpha}^{(2)}$ (and likewise for their ambient counterparts). The claim follows from Definition 2.17. \square

It is straightforward to see that our graded decompositions agree with those conjectured in [HKO⁺02b, App. A], where the $r = 2$ case is similar to the proof of Proposition 9.12.

9.6. Type $F_4^{(1)}$. We can also describe the set $\text{RC}^{HW}(B^{r,s})$ for $r = 1, 2, 4$ in type $F_4^{(1)}$.

For $r = 1, 2$ in type $F_4^{(1)}$, this is the same as for type $E_6^{(2)}$ except for $\nu^{(3)}$ and $\nu^{(4)}$ are scaled by 2. This follows from Definition 2.17, that $\gamma_{0,1,2} = 2$, and that $\gamma_{3,4} = 1$.

Proposition 9.32. *Consider the KR crystal $B^{4,s}$ of type $F_4^{(1)}$. We have*

$$\text{RC}(B^{4,s}) = \bigoplus_{\substack{k_1, k_2 \geq 0 \\ k_1 + k_2 \leq s/2}} \text{RC}(B^{4,s}; k_2 \overline{\Lambda}_1 + (s - 2k_1 - 2k_2) \overline{\Lambda}_4).$$

Moreover the highest weight rigged configurations in $\text{RC}(B^{4,s})$ are given by

$$\begin{aligned} \nu^{(1)} &= (k_1, k_1) \\ \nu^{(2)} &= (k_1 + k_2, k_1, k_1, k_1), \\ \nu^{(3)} &= (2k_1 + 2k_2, 2k_1, 2k_1), \\ \nu^{(4)} &= (2k_1 + 2k_2, 2k_1), \end{aligned}$$

with all riggings 0.

Proof. Similar to the proof of Proposition 9.31 except for we select all nodes at even depths and the note above about devirtualization. \square

Given the devirtualization map and that all nodes for $r = 1, 2, 4$ appear at even levels in the ambient Kleber tree, and Proposition 9.12, it is straightforward to see that our crystals agree with the conjectured decompositions of [HKO⁺99, App. A].

10. OUTLOOK

For untwisted non-simply-laced affine types, the author believes that there is a way to modify the description of the rigged configurations such that each partition is scaled by $1/T_r$ as in Remark 6.2. In particular, this modification will be similar to the description given here and the proof would be similar and uniform. We note that this was done for type $C_n^{(1)}$ and $B_n^{(1)}$ for $B^{1,1}$ in [OSS03a]. However, we need to take $B^{1,2}$ for type $C_n^{(1)}$ and $B^{2,1}$ for type $B_n^{(1)}$ in order to get the adjoint node (and have a perfect crystal of level 1). As such, a modification to our description would be needed. It seems plausible that the extra possibility of case (Q), a singular row of length one less than previously selected, as given in δ_1 of type $B_n^{(1)}$ from [OSS03a] is the necessary modification. For type $G_2^{(1)}$, this is an open problem from [Scr16], which we give here in more generality.

Problem 10.1. Describe explicitly the map δ_θ for \mathfrak{g} of untwisted non-simply-laced affine type.

APPENDIX A. KR TABLEAUX FOR FUNDAMENTAL WEIGHTS

In this section, we list the classically highest weight KR tableaux for $B^{r,1}$ for types $E_{6,7,8}^{(1)}$, $E_6^{(2)}$, $F_4^{(1)}$, and $G_2^{(1)}$.

Recall that highest weight rigged configurations must have $0 \leq p_i^{(a)}$ for all $(a, i) \in \mathcal{H}_0$. Moreover, all rigged configurations in this section will have $p_i^{(a)} = 0$ except for possibly one $(b, j) \in \mathcal{H}_0$ where $p_j^{(b)} = 1$. Therefore we describe the rigged configuration simply by its configuration ν and if $p_i^{(a)} = 1$, then we write the corresponding rigging x as the subscript (x) . We also write the column tableau $\begin{bmatrix} x_1 & x_2 & x_3 & \cdots & x_r \end{bmatrix}^t$ as $[x_1, x_2, x_3, \dots, x_r]$.

Example A.1. In type $E_7^{(1)}$ for $B^{4,1}$, we denote the rigged configuration

$$0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \quad 1 \begin{array}{|c|} \hline \square \\ \hline \end{array} 1 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 \quad 0 \begin{array}{|c|} \hline \square \\ \hline \end{array} 0 ,$$

$$\begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array} \quad \begin{array}{|c|} \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline 0 \\ \hline \end{array}$$

by $(1, 11, 1_{(1)}1_{(0)}, 1111, 111, 11, 1)$ or more compactly $(1, 1^2, 1_{(1,0)}^2, 1^4, 1^3, 1^2, 1)$.

In this remainder section, we give the KR tableaux for $B^{r,1}$ for the exceptional types (except for $r = 4, 5$ in type $E_8^{(1)}$, which can be generated using SAGEMATH [Dev17]).

A.1. **Type $E_6^{(1)}$.** $r = 1$

$$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) \mapsto [1]$$

$r = 2$

$$\begin{aligned} (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) &\mapsto [1, \overline{1}6, 2\overline{6}] \\ (1, 1^2, 1^2, 1^3, 1^2, 1) &\mapsto [1, \overline{1}6, \overline{6}] \end{aligned}$$

$r = 3$

$$\begin{aligned} (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) &\mapsto [1, \overline{1}3] \\ (1, 1, 1^2, 1^2, 1, \emptyset) &\mapsto [1, \overline{1}6] \end{aligned}$$

$r = 4$

$$\begin{aligned} (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) &\mapsto [1, \overline{1}3, \overline{3}4] \\ (\emptyset, 1, 1, 1^2, 1, \emptyset) &\mapsto [1, \overline{1}3, 1\overline{3}6] \\ (1, 1_{(0)}, 1, 1^3, 1^2, 1) &\mapsto [1, \overline{1}6, 2\overline{6}] \\ (1, 1_{(1)}, 1^2, 1^3, 1^2, 1) &\mapsto [1, \overline{1}3, 2\overline{3}] \\ (1^2, 1^3, 1^4, 1^6, 1^4, 1^2) &\mapsto [1, \overline{1}6, \overline{6}] \end{aligned}$$

$r = 5$

$$\begin{aligned} (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) &\mapsto [1, \overline{1}6, 2\overline{6}, \overline{2}5] \\ (\emptyset, 1, 1, 1^2, 1^2, 1) &\mapsto [1, \overline{1}6, 2\overline{6}, 1\overline{2}] \end{aligned}$$

$r = 6$

$$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) \mapsto [1, \overline{1}6]$$

A.2. **Type $E_7^{(1)}$.** $r = 1$

$$\begin{aligned} (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) &\mapsto [7, 1\overline{7}] \\ (1^2, 1^2, 1^3, 1^4, 1^3, 1^2, 1) &\mapsto [7, \overline{7}] \end{aligned}$$

$r = 2$

$$\begin{aligned} (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) &\mapsto [7, 1\overline{7}, \overline{1}2] \\ (1, 1^2, 1^2, 1^3, 1^2, 1, \emptyset) &\mapsto [7, 1\overline{7}, \overline{1}7] \end{aligned}$$

$r = 3$

$$\begin{aligned} (\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset) &\mapsto [7, 1\overline{7}, \overline{1}2, \overline{2}3] \\ (1, 1, 1^2, 1^2, 1, \emptyset, \emptyset) &\mapsto [7, 1\overline{7}, \overline{1}2, \overline{2}6] \\ (1_{(0)}, 1^2, 1^3, 1^4, 1^3, 1^2, 1) &\mapsto [7, 1\overline{7}, \overline{1}7, 1\overline{7}] \\ (1_{(1)}, 1^2, 1^3, 1^4, 1^3, 1^2, 1) &\mapsto [7, 1\overline{7}, \overline{1}2, \overline{1}2] \\ (1^3, 1^4, 1^6, 1^8, 1^6, 1^4, 1^2) &\mapsto [7, 1\overline{7}, \overline{1}7, \overline{7}] \end{aligned}$$

$r = 4$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[7, 6\overline{7}, 5\overline{6}, 4\overline{5}]$
$(\emptyset, 1, 1, 1^2, 1, \emptyset, \emptyset)$	\mapsto	$[7, 6\overline{7}, 5\overline{6}, 1\overline{5}6]$
$(\emptyset, 1^2, 1^2, 1^4, 1^3, 1^2, 1)$	\mapsto	$[7, 6\overline{7}, 1\overline{6}7, 1\overline{7}]$
$(1, 1_{(0)}, 1^2, 1^3, 1^2, 1, \emptyset)$	\mapsto	$[7, 6\overline{7}, 1\overline{6}7, \overline{1}2]$
$(1, 1_{(1)}, 1^2, 1^3, 1^2, 1, \emptyset)$	\mapsto	$[7, 6\overline{7}, 5\overline{6}, 2\overline{5}7]$
$(1, 1^2, 1^2_{(0,0)}, 1^4, 1^3, 1^2, 1)$	\mapsto	$[7, 1\overline{7}, \overline{1}2, \overline{2}3]$
$(1, 1^2, 1^2_{(1,0)}, 1^4, 1^3, 1^2, 1)$	\mapsto	$[7, 6\overline{7}, 1\overline{6}7, \overline{1}3\overline{7}]$
$(1, 1^2, 1^2_{(1,1)}, 1^4, 1^3, 1^2, 1)$	\mapsto	$[7, 6\overline{7}, 5\overline{6}, 3\overline{5}]$
$(1^2, 1^3, 1^4, 1^6, 1^4, 1^2, \emptyset)$	\mapsto	$[7, 6\overline{7}, 1\overline{6}7, \overline{1}7]$
$(1^2, 1^3, 1^4, 1^6, 1^4, 1^2_{(0,0)}, 1)$	\mapsto	$[7, 1\overline{7}, \overline{1}2, \overline{2}6]$
$(1^2, 1^3, 1^4, 1^6, 1^4, 1^2_{(1,0)}, 1)$	\mapsto	$[7, 6\overline{7}, 1\overline{6}7, \overline{1}6\overline{7}]$
$(1^2, 1^3, 1^4, 1^6, 1^4, 1^2_{(1,1)}, 1)$	\mapsto	$[7, 6\overline{7}, 5\overline{6}, \overline{5}6]$
$(1^2_{(0,0)}, 1^4, 1^5, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[7, 1\overline{7}, \overline{1}7, 1\overline{7}]$
$(1^2_{(1,0)}, 1^4, 1^5, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[7, 1\overline{7}, \overline{1}2, 1\overline{2}]$
$(1^2_{(1,1)}, 1^4, 1^5, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[7, 6\overline{7}, 1\overline{6}7, \overline{7}]$
$(1^4, 1^6, 1^8, 1^{12}, 1^9, 1^6, 1^3)$	\mapsto	$[7, 1\overline{7}, \overline{1}7, \overline{7}]$
$(2, 21, 2^2, 2^21^2, 21^2, 1^2, 1)$	\mapsto	$[7, 6\overline{7}, 2\overline{6}, \overline{2}6]$
$(1^2, 21^2, 21^3, 2^21^4, 2^21^2, 2^2, 2)$	\mapsto	$[7, 6\overline{7}, 2\overline{6}, \overline{2}1]$
$(2^2, 2^21^2, 2^31^2, 2^41^4, 2^31^3, 2^21^2, 21)$	\mapsto	$[7, 6\overline{7}, \overline{6}7, \overline{7}]$

$r = 5$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[7, 6\overline{7}, 5\overline{6}]$
$(\emptyset, 1, 1, 1^2, 1^2, 1, \emptyset)$	\mapsto	$[7, 6\overline{7}, 1\overline{6}7]$
$(1, 1_{(0)}, 1^2, 1^3, 1^3, 1^2, 1)$	\mapsto	$[7, 1\overline{7}, \overline{1}2]$
$(1, 1_{(1)}, 1^2, 1^3, 1^3, 1^2, 1)$	\mapsto	$[7, 6\overline{7}, 2\overline{6}]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^3, 1_{(0)})$	\mapsto	$[7, 1\overline{7}, \overline{1}7]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^3, 1_{(1)})$	\mapsto	$[7, 6\overline{7}, \overline{6}7]$

$r = 6$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[7, 6\overline{7}]$
$(\emptyset, 1, 1, 1^2, 1^2, 1^2, 1)$	\mapsto	$[7, 1\overline{7}]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^2)$	\mapsto	$[7, \overline{7}]$

$r = 7$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[7]$
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A.3. Type $E_8^{(1)}$. $r = 1$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[8, 1\overline{8}]$
$(1^2, 1^2, 1^3, 1^4, 1^3, 1^2, 1, \emptyset)$	\mapsto	$[8, \emptyset]$
$(1^4, 1^5, 1^7, 1^{10}, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[\emptyset, \emptyset]$

$r = 2$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[8, 1\overline{8}, \overline{1}2]$
$(1, 1^2, 1^2, 1^3, 1^2, 1, \emptyset, \emptyset)$	\mapsto	$[8, 1\overline{8}, \overline{1}7]$
$(1_{(0)}, 1^3, 1^3, 1^5, 1^4, 1^3, 1^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}1]$
$(1_{(1)}, 1^3, 1^3, 1^5, 1^4, 1^3, 1^2, 1)$	\mapsto	$[8, 1\overline{8}, \emptyset]$
$(1^3, 1^5, 1^6, 1^9, 1^7, 1^5, 1^3, 1_{(0)})$	\mapsto	$[8, \emptyset, 8\overline{8}]$
$(1^3, 1^5, 1^6, 1^9, 1^7, 1^5, 1^3, 1_{(1)})$	\mapsto	$[8, \emptyset, \emptyset]$
$(1^5, 1^8, 1^{10}, 1^{15}, 1^{12}, 1^9, 1^6, 1^3)$	\mapsto	$[\emptyset, \emptyset, \emptyset]$

$r = 3$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[8, 1\overline{8}, \overline{1}2, \overline{2}3]$
$(1, 1, 1^2, 1^2, 1, \emptyset, \emptyset, \emptyset)$	\mapsto	$[8, 1\overline{8}, \overline{1}2, \overline{2}6]$
$(1_{(0)}, 1^2, 1^3, 1^4, 1^3, 1^2, 1, \emptyset)$	\mapsto	$[8, 1\overline{8}, \overline{1}7, 1\overline{7}8]$
$(1_{(1)}, 1^2, 1^3, 1^4, 1^3, 1^2, 1, \emptyset)$	\mapsto	$[8, 1\overline{8}, \overline{1}2, 1\overline{2}8]$
$(1^2, 1_{(0,0)}^2, 1^4, 1^5, 1^4, 1^3, 1^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}7, 2\overline{7}]$
$(1^2, 1_{(1,0)}^2, 1^4, 1^5, 1^4, 1^3, 1^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}2, \overline{2}2]$
$(1^2, 1_{(1,1)}^2, 1^4, 1^5, 1^4, 1^3, 1^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}2, \emptyset]$
$(1^3, 1^4, 1^6, 1^8, 1^6, 1^4, 1^2, \emptyset)$	\mapsto	$[8, 1\overline{8}, \overline{1}7, \overline{7}88]$
$(1^3, 1^4, 1^6, 1^8, 1^6, 1^4, 1_{(0,0)}^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}1, \overline{1}7]$
$(1^3, 1^4, 1^6, 1^8, 1^6, 1^4, 1_{(1,0)}^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}7, \overline{7}7]$
$(1^3, 1^4, 1^6, 1^8, 1^6, 1^4, 1_{(1,1)}^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}7, \emptyset]$
$(1_{(0,0,0)}^3, 1^5, 1^7, 1^{10}, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[8, \emptyset, \overline{8}8, 1\overline{8}]$
$(1_{(1,0,0)}^3, 1^5, 1^7, 1^{10}, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[8, 1\overline{8}, \overline{1}1, \overline{1}1]$
$(1_{(1,1,0)}^3, 1^5, 1^7, 1^{10}, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[8, 1\overline{8}, \overline{1}1, \emptyset]$
$(1_{(1,1,1)}^3, 1^5, 1^7, 1^{10}, 1^8, 1^6, 1^4, 1^2)$	\mapsto	$[8, 1\overline{8}, \emptyset, \emptyset]$
$(1^5, 1^7, 1^{10}, 1^{14}, 1^{11}, 1^8, 1^5, 1_{(0,0)}^2)$	\mapsto	$[8, \emptyset, \overline{8}8, \overline{8}8]$
$(1^5, 1^7, 1^{10}, 1^{14}, 1^{11}, 1^8, 1^5, 1_{(1,0)}^2)$	\mapsto	$[8, \emptyset, \overline{8}8, \emptyset]$
$(1^5, 1^7, 1^{10}, 1^{14}, 1^{11}, 1^8, 1^5, 1_{(1,1)}^2)$	\mapsto	$[8, \emptyset, \emptyset, \emptyset]$
$(1^7, 1^{10}, 1^{14}, 1^{20}, 1^{16}, 1^{12}, 1^8, 1^4)$	\mapsto	$[\emptyset, \emptyset, \emptyset, \emptyset]$
$(21, 2^2, 2^2 1^2, 2^3 1^2, 2^2 1^2, 21^2, 1^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}2, \overline{2}7]$
$(1^3, 21^3, 21^5, 2^2 1^6, 2^2 1^4, 2^2 1^2, 2^2, 2)$	\mapsto	$[8, 1\overline{8}, \overline{1}7, 1\overline{7}]$
$(2^2 1_{(0)}, 2^2 1^3, 2^3 1^4, 2^4 1^6, 2^3 1^5, 2^2 1^4, 21^3, 1^2)$	\mapsto	$[8, 1\overline{8}, \overline{1}1, \overline{1}8]$
$(2^2 1_{(1)}, 2^2 1^3, 2^3 1^4, 2^4 1^6, 2^3 1^5, 2^2 1^4, 21^3, 1^2)$	\mapsto	$[8, 1\overline{8}, \emptyset, \overline{1}8]$
$(2^2 1^3, 2^3 1^4, 2^4 1^6, 2^6 1^8, 2^5 1^6, 2^4 1^4, 2^3 1^2, 2^2)$	\mapsto	$[8, \emptyset, \emptyset, \overline{8}]$

$r = 6$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[8, 7\overline{8}, 6\overline{7}]$
$(\emptyset, 1, 1, 1^2, 1^2, 1^1, 1, \emptyset)$	\mapsto	$[8, 7\overline{8}, 1\overline{7}8]$
$(1, 1_{(0)}, 1^2, 1^3, 1^3, 1^3, 1^2, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}2]$
$(1, 1_{(1)}, 1^2, 1^3, 1^3, 1^3, 1^2, 1)$	\mapsto	$[8, 7\overline{8}, 2\overline{7}]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^2, \emptyset)$	\mapsto	$[8, 7\overline{8}, \overline{7}88]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^2_{(0,0)}, 1)$	\mapsto	$[8, 1\overline{8}, \overline{1}7]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^2_{(1,0)}, 1)$	\mapsto	$[8, 7\overline{8}, \overline{7}7]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^2_{(1,1)}, 1)$	\mapsto	$[8, 7\overline{8}, \emptyset]$
$(1^2_{(0,0)}, 1^4, 1^5, 1^8, 1^7, 1^6, 1^4, 1^2)$	\mapsto	$[8, \overline{8}8, 1\overline{8}]$
$(1^2_{(1,0)}, 1^4, 1^5, 1^8, 1^7, 1^6, 1^4, 1^2)$	\mapsto	$[8, 1\overline{8}, \overline{1}1]$
$(1^2_{(1,1)}, 1^4, 1^5, 1^8, 1^7, 1^6, 1^4, 1^2)$	\mapsto	$[8, 1\overline{8}8, \emptyset]$
$(1^4, 1^6, 1^8, 1^{12}, 1^{10}, 1^8, 1^5, 1^2_{(0,0)})$	\mapsto	$[8, \overline{8}8, \overline{8}8]$
$(1^4, 1^6, 1^8, 1^{12}, 1^{10}, 1^8, 1^5, 1^2_{(1,0)})$	\mapsto	$[8, \overline{8}8, \emptyset]$
$(1^4, 1^6, 1^8, 1^{12}, 1^{10}, 1^8, 1^5, 1^2_{(1,1)})$	\mapsto	$[8, \emptyset, \emptyset]$
$(1^6, 1^9, 1^{12}, 1^{18}, 1^{15}, 1^{12}, 1^8, 1^4)$	\mapsto	$[\emptyset, \emptyset, \emptyset]$
$(1^2, 21^2, 21^3, 2^21^4, 2^21^3, 2^21^2, 2^2, 2)$	\mapsto	$[8, 7\overline{8}, 1\overline{7}]$
$(2^2, 2^21^2, 2^31^2, 2^41^4, 2^31^4, 2^21^4, 21^3, 1^2)$	\mapsto	$[8, 1\overline{8}, \overline{1}8]$
$(2^21^2, 2^31^3, 2^41^4, 2^61^6, 2^51^5, 2^41^4, 2^31^2, 2^2)$	\mapsto	$[8, \emptyset, \overline{8}]$

$r = 7$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[8, 7\overline{8}]$
$(\emptyset, 1, 1, 1^2, 1^2, 1^2, 1^2, 1)$	\mapsto	$[8, 1\overline{8}]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^3, 1_{(0)})$	\mapsto	$[8, 8\overline{8}]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^3, 1_{(1)})$	\mapsto	$[8, \emptyset]$
$(1^4, 1^6, 1^8, 1^{12}, 1^{10}, 1^8, 1^6, 1^3)$	\mapsto	$[\emptyset, \emptyset]$

$r = 8$

$(\emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[8]$
$(1^2, 1^3, 1^4, 1^6, 1^5, 1^4, 1^3, 1^2)$	\mapsto	$[\emptyset]$

A.4. **Type** $E_6^{(2)}$. $r = 1$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[1]$
$(1^2, 1^3, 1^2, 1)$	\mapsto	$[\emptyset]$

$r = 2$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[1, \overline{12}]$
$(1, 1^2, 1, \emptyset)$	\mapsto	$[1, \overline{14}]$
$(1_{(0)}, 1^3, 1^2, 1)$	\mapsto	$[1, \overline{11}]$
$(1_{(1)}, 1^3, 1^2, 1)$	\mapsto	$[1, \emptyset]$
$(1^3, 1^6, 1^4, 1^2)$	\mapsto	$[\emptyset, \emptyset]$

$r = 3$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[1, \overline{12}, \overline{23}]$
$(\emptyset, 1, 1, \emptyset)$	\mapsto	$[1, \overline{12}, 1\overline{24}]$
$(\emptyset, 1^2, 1^2, 1)$	\mapsto	$[1, \overline{12}, 11\overline{2}]$
$(1, 1_{(0,0)}^2, 1^2, 1)$	\mapsto	$[1, \overline{14}, 2\overline{4}]$
$(1, 1_{(1,0)}^2, 1^2, 1)$	\mapsto	$[1, \overline{12}, \overline{22}]$
$(1, 1_{(1,1)}^2, 1^2, 1)$	\mapsto	$[1, \overline{12}, \emptyset]$
$(1^2, 1^4, 1^3, 1_{(0)})$	\mapsto	$[1, \emptyset, \overline{14}]$
$(1^2, 1^4, 1^3, 1_{(1)})$	\mapsto	$[1, \overline{14}, \emptyset]$
$(1_{(0,0)}^2, 1^5, 1^4, 1^2)$	\mapsto	$[1, \overline{11}, \overline{11}]$
$(1_{(1,0)}^2, 1^5, 1^4, 1^2)$	\mapsto	$[1, \overline{11}, \emptyset]$
$(1_{(1,1)}^2, 1^5, 1^4, 1^2)$	\mapsto	$[1, \emptyset, \emptyset]$
$(1^4, 1^8, 1^6, 1^3)$	\mapsto	$[\emptyset, \emptyset, \emptyset]$
$(2, 2^2, 21, 1)$	\mapsto	$[1, \overline{12}, \overline{24}]$
$(1^2, 21^3, 21^2, 2)$	\mapsto	$[1, \overline{14}, 1\overline{4}]$
$(2^2, 2^3 1^2, 2^2 1^2, 21)$	\mapsto	$[1, \emptyset, \overline{1}]$

$r = 4$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[1, \overline{14}]$
$(\emptyset, 1, 1, 1)$	\mapsto	$[1, \emptyset]$
$(1^2, 1^4, 1^3, 1^2)$	\mapsto	$[\emptyset, \emptyset]$

A.5. **Type $F_4^{(1)}$.** We follow Proposition 4.2 to describe the elements of $B(\overline{\Lambda}_4)$.
 $r = 1$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[4, 1\overline{4}]$
$(1^2, 1^3, 2^2, 2)$	\mapsto	$[4, \overline{4}]$

$r = 2$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[4, 3\overline{4}, 2\overline{3}]$
$(1, 1^2, 2, \emptyset)$	\mapsto	$[4, 3\overline{4}, \overline{344}]$
$(1_{(0)}, 1^3, 2^2, 2)$	\mapsto	$[4, 4\overline{4}, 1\overline{4}]$
$(1_{(1)}, 1^3, 2^2, 2)$	\mapsto	$[4, 3\overline{4}, 1\overline{3}]$
$(1^3, 1^6, 2^4, 2^2)$	\mapsto	$[4, 4\overline{4}, \overline{4}]$

$r = 3$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[4, 3\overline{4}]$
$(1, 1^2, 21, 1)$	\mapsto	$[4, 4\overline{4}]$

$r = 4$

$(\emptyset, \emptyset, \emptyset, \emptyset)$	\mapsto	$[4]$
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A.6. Type $G_2^{(1)}$. $r = 1$

(\emptyset, \emptyset)	\mapsto	$[1]$
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$r = 2$

(\emptyset, \emptyset)	\mapsto	$[1, 2]$
$(3, 1^2)$	\mapsto	$[1, \overline{1}]$

APPENDIX B. EXAMPLES WITH SAGE

We give some examples using Sage [Dev17], which has been implemented by the author.
We first construct Example 4.3.

```
sage: RC = RiggedConfigurations(['E', 6, 2], [[1, 1]]*4)
sage: mg = RC(partition_list=[[2, 2, 1, 1], [2, 2, 2, 1, 1, 1], [2, 2, 1, 1], [2, 1]],
....:         rigging_list=[[1, 0, 2, 1], [0, 0, 0, 0, 0, 0], [0, 0, 0, 0], [0, 0]])
sage: ascii_art(mg)
1[ ] 1[ ] 0[ ] 0[ ] 0[ ] 0[ ]
1[ ] 0[ ] 0[ ] 0[ ] 0[ ] 0[ ]
2[ ] 2[ ] 0[ ] 0[ ] 0[ ] 0[ ]
2[ ] 1[ ] 0[ ] 0[ ] 0[ ] 0[ ]
      0[ ] 0[ ]
      0[ ] 0[ ]
sage: mg.to_tensor_product_of_kirillov_reshetikhin_tableaux().pp()
(1, -2) (X) (-1, 2) (X) E (X) (1,)
```

Next, we construct Example 5.2.

```
sage: RC = RiggedConfigurations(['E', 7, 1], [[4, 1]])
sage: mg = RC.module_generators[6]
sage: ascii_art(mg)
0[ ] 0[ ] 0[ ] 1[ ] 0[ ] 0[ ] 0[ ] 0[ ]
      0[ ] 0[ ] 1[ ] 0[ ] 0[ ] 0[ ] 0[ ]
              0[ ] 0[ ] 0[ ]
              0[ ] 0[ ]
sage: mg.to_tensor_product_of_kirillov_reshetikhin_tableaux().pp()
      (7,)
(-7, 6)
```

$(-6, 7, 1)$ $(-1, -7, 3)$

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